# Redundant Matrices for Linear Transformations

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**Abstract**

Matrices for linear transformations with respect to a spanning set, rather than a basis, are introduced and are shown to have properties that reflect those of the transformation. Specifically, it is shown that there is an invariant subspace for the matrix on which it is isomorphic to the transformation. In particular, all eigenvalues of the transforma- tion are eigenvalues of the matrix. This construct has been used to find the spectrum of composition operators on a Hilbert space where a natural spanning set exists that is not a basis.

## 1 Introduction

One of the first things we learn about linear transformations is how to represent them as matrices. We find that, for a specified basis, the representing matrix is unique and that the transformation in the algebra of linear transformations is isomorphic to its representing matrix in the algebra of matrices. In this paper, we wish to generalize this notion by beginning with a spanning set that is not necessarily a basis: if $\{v_1, v_2, \cdots, v_n\}$ spans the vector space $\mathcal{V}$ and $T$ is a linear transformation of $\mathcal{V}$ into $\mathcal{V}$, the matrix $A = (a_{ij})$ *is a redundant matrix for the transformation* $T$ *with respect to this spanning set* if the matrix entries satisfy

$$Tv_j = a_{1j}v_1 + a_{2j}v_2 + \cdots + a_{nj}v_n$$

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If the spanning set is not a basis, then the vectors are linearly dependent and there are many redundant matrices for the transformation with respect to the given spanning set. Moreover, since the dimension of $\mathcal{V}$ is less than $n$, we know that $T$ in the algebra of linear transformations on $\mathcal{V}$ cannot possibly be isomorphic to $A$ in the algebra of $n \times n$ matrices. In this paper, then, we will explore the relationship between $A$ and $T$. We will show that there is a subspace of $\mathbf{C}^n$ that is invariant under $A$ such that the restriction of $A$ to this subspace is isomorphic to $T$. In particular, this means that the every eigenvalue of $T$ is also an eigenvalue of $A$.

Since it may seem that this is a sterile generalization, we should point out the situation in which this question arose. In the study of composition operators on Hilbert spaces of analytic functions, there are vectors associated with derivatives of the functions at various points and some operators have natural expressions in terms of these functions. For example, if we are considering a space of functions analytic on the unit ball in $\mathbf{C}^2$, the vectors $D_1$ and $D_2$ are the vectors so that

$$\langle f, D_1 \rangle = \frac{\partial f}{\partial z_1}(0) \quad \text{and} \quad \langle f, D_2 \rangle = \frac{\partial f}{\partial z_2}(0)$$

where $f$ is a function in the space and $\langle \cdot, \cdot \rangle$ is the inner product on the Hilbert space. The transformation we considered, $C_\varphi^*$ restricted to an invariant subspace, has a natural expression in terms of these functions. Similarly, there are vectors $D_{11}$, $D_{12}$, $D_{21}$, and $D_{22}$ and $C_\varphi^*$ restricted to another invariant subspace has a natural expression involving these vectors. The difficulty arises that, because of equality of mixed partials, $D_{12} = D_{21}$ so the natural vectors for the problem are not a basis, rather a spanning set. In the attack on their problem, Cowen and MacCluer [1, page 271–275] proved the spectral inclusion property mentioned above and used it to compute the spectrum of $C_\varphi$.

In the next section, we introduce the representing operator for a spanning set and we show that a redundant matrix for a transformation can be used in much the same way that a matrix with respect to a basis can be used. The following section contains the proof of the main result and an example that illustrates the essential difficulty. The final section gives an application to symmetric tensor products that generalizes the application above.

## 2 Basic Ideas

Suppose $T$ is a linear transformation of the vector space $\mathcal{V}$ into itself. Let $\{v_1, v_2, \cdots, v_n\}$ span $\mathcal{V}$ and suppose the matrix $A = (a_{ij})$ is a redundant matrix for the transformation $T$ with respect to this spanning set, that is, suppose the matrix entries satisfy

$$Tv_j = a_{1j}v_1 + a_{2j}v_2 + \cdots + a_{nj}v_n$$

for each vector of the spanning set. If $w$ is a vector of $\mathcal{V}$ that is written as a linear combination of the spanning vectors, then $Tw$ can be "computed" by using $A$. We formalize this statement using the representing operator.

**Definition** The *representing operator $R$ for the spanning set $\{v_1, v_2, \cdots, v_n\}$ of $\mathcal{V}$*, is the linear operator $R\colon \mathbf{C}^n \mapsto \mathcal{V}$ given by

$$R(\alpha_1, \cdots, \alpha_n) = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

**Theorem 1** *If $A$ is a redundant matrix for the linear transformation $T$ with respect to the spanning set $\{v_1, v_2, \cdots, v_n\}$ for $\mathcal{V}$ and $R$ is the representing operator for this spanning set, then $TR = RA$. Conversely, suppose $T$ is a linear transformation on $\mathcal{V}$ and $R$ is the representing operator for the spanning set $\{v_1, v_2, \cdots, v_n\}$ on $\mathcal{V}$. If $A$ is an $n \times n$ matrix such that $TR = RA$, then $A$ is a redundant matrix for $T$.*

**Proof.** Let $x = (\alpha_1, \cdots, \alpha_n)$ be in $\mathbf{C}^n$ and $w = Rx = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Then

$$\begin{aligned}
TRx &= Tw = \alpha_1 Tv_1 + \cdots + \alpha_n Tv_n \\
&= \alpha_1 (a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n) \\
&\quad +\alpha_2 (a_{12}v_1 + a_{22}v_2 + \cdots + a_{n2}v_n) \\
&\quad + \cdots + \alpha_n (a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{nn}v_n) \\
&= (a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n)\, v_1 \\
&\quad + (a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n)\, v_2 \\
&\quad + \cdots + (a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n)\, v_n \\
&= RAx
\end{aligned}$$

Since this is true for all vectors in $\mathbf{C}^n$, the conclusion holds.

The converse follows from applying the relation $TR = RA$ and the definition of representing operator to the standard basis vectors for $\mathbf{C}^n$. ■

It follows that algebraic manipulations of $A$ correspond closely to the same manipulations of $T$.

**Corollary 2** *If $A$ is a redundant matrix for $T$ with respect to a particular spanning set, then for any polynomial $p$, the matrix $p(A)$ is a redundant matrix for $p(T)$ with respect to the same spanning set. Moreover, if $A$ is invertible, then $T$ is invertible and $A^{-1}$ is a redundant matrix for $T^{-1}$ for this spanning set.*

**Proof.** Since $RA = TR$, we have

$$(T - I_{\mathcal{V}})R = TR - I_{\mathcal{V}}R = TR - R = RA - RI_n = R(A - I_n)$$

We also have

$$RA^2 = (RA)A = (TR)A = T(RA) = T(TR) = T^2R$$

It follows in the same way that $RA^k = T^kR$ and, therefore, that $Rp(A) = p(T)R$ for any polynomial $p$. By Theorem 1, this means $p(A)$ is a redundant matrix for $p(T)$.

Suppose $A$ is invertible, then

$$R = R(AA^{-1}) = (RA)A^{-1} = (TR)A^{-1} = T(RA^{-1})$$

Since the range of $R$ is $\mathcal{V}$, the range of $T$ is also $\mathcal{V}$ and $T$ is invertible. Multiplying both sides of the above equation by $T^{-1}$ shows $T^{-1}R = RA^{-1}$ so Theorem 1 shows $A^{-1}$ is a redundant matrix for $T^{-1}$ with respect to the given spanning set. ■

Since $\{v_1, v_2, \cdots, v_n\}$ is a spanning set for $\mathcal{V}$, the representing operator is a map of $\mathbf{C}^n$ onto $\mathcal{V}$. It follows that there is a map $S$ of $\mathcal{V}$ into $\mathbf{C}^n$ so that $RS = I_{\mathcal{V}}$, indeed, there will be many such right inverses if $\mathcal{V}$ is not $n$–dimensional.

If $R$ is a representing operator and $S$ is a right inverse for it, one way in which redundant matrices can arise is by taking $A = STR$: clearly, in this case, $RA = R(STR) = (RS)(TR) = ITR = TR$, so $A$ is a representing matrix by Theorem 1 above. However, not all such redundant matrices for $T$ arise in this way.

For example, consider the linear transformation on $\mathbf{C}^2$ defined by $T(a,b) = (4a+2b, -a+b)$ and take $v_1 = (1,1)$, $v_2 = (1,2)$, and $v_3 = (2,1)$ be the spanning set for $\mathbf{C}^2$. Since $Tv_1 = (6,0) = -3v_1 - v_2 + 5v_3$, $Tv_2 = (8,1) = 3v_1 - 3v_2 + 4v_3$, and $Tv_3 = (10,-1) = 6v_1 - 6v_2 + 5v_3$, the matrix

$$A = \begin{pmatrix} -3 & 3 & 6 \\ -1 & -3 & -6 \\ 5 & 4 & 5 \end{pmatrix}$$

is a redundant matrix for $T$ with respect to this spanning set. If we choose the usual basis $e_1 = (1,0)$ and $e_2 = (0,1)$ for $\mathbf{C}^2$, then the matrices for $T$ and $R$ are (abusing notation somewhat)

$$T = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

and the intertwining relation is easily checked

$$TR = \begin{pmatrix} 6 & 8 & 10 \\ 0 & 1 & -1 \end{pmatrix} = RA$$

If there were a right inverse $S$ for $R$ so that $A = STR$, then, because $T$ has rank 2, the rank of $A$ would be no more than 2. However, the rank of $A$ is easily checked to be 3, so $A \neq STR$ for any $S$!

The eigenvalues of $T$ are 2 and 3; we will show in the next section that the fact that they are also eigenvalues of $A$ is not a coincidence.

# 3 The Main Theorem

**Theorem 3** *If $A$ is a redundant matrix for the linear transformation $T$, there is a subspace $M$ of $\mathbf{C}^n$ such that $M$ is invariant under $A$ and the restriction of $A$ to $M$ is isomorphic to $T$. Conversely, if $T$ is a linear transformation on $\mathcal{V}$ and $A$ is an $n \times n$ matrix with an invariant subspace $M$ such that the restriction of $A$ to $M$ is isomorphic to $T$, then $A$ is a redundant matrix for $T$.*

The proof of the main theorem uses the representing operator $R$ to relate the Jordan structures of $A$ and $T$. Before proving the theorem, we recall some terminology and prove two lemmas having to do with the Jordan structures of $T$ and $A$.

We will call a non-zero vector $u$ a *generalized eigenvector for $P$ corresponding to the eigenvalue $\lambda$* if there is an integer $k$ so that $(P-\lambda I)^k u = 0$ and in this case, we say the *index of $u$ is $k$* if $(P-\lambda I)^{k-1}u \neq 0$. We will say the set $\{u_1, u_2, \cdots, u_k\}$ is a *Jordan chain* for the eigenvalue $\lambda$ if $(P-\lambda I)u_k = u_{k-1}$, $(P-\lambda I)u_{k-1} = u_{k-2}$, $\cdots$, $(P-\lambda I)u_2 = u_1$, and $(P-\lambda I)u_1 = 0$.

**Lemma 4** *Suppose $x$ is a generalized eigenvector for $A$ corresponding to the eigenvalue $\lambda$. Then either $Rx$ is a generalized eigenvector for $T$ corresponding to $\lambda$ or $Rx = 0$. Moreover, the index of $x$ as a generalized eigenvector is no less than the index of $Rx$.*

**Proof.** Let $k$ be a positive integer so that

$$(A-\lambda)^k x = 0 \quad \text{and} \quad (A-\lambda)^{k-1}x \neq 0$$

As we noted after Theorem 1, since $RA = TR$, it follows that

$$(T-\lambda)^k(Rx) = R(A-\lambda)^k x = R0 = 0$$

so either $Rx$ is a generalized eigenvector for $T$ corresponding to $\lambda$ (with index less than or equal to $k$) or $Rx = 0$, as we were to prove. ■

**Corollary 5** *If $A$ is a redundant matrix for $T$ and $A$ is diagonalizable, then $T$ is diagonalizable.*

**Proof.** Since $A$ is diagonalizable, $\mathbf{C}^n$ is spanned by eigenvectors of $A$. The representing matrix $R$ maps $\mathbf{C}^n$ onto $\mathcal{V}$ so it takes this spanning set of eigenvectors onto a spanning set for $\mathcal{V}$. By Lemma 4, the image vectors are either eigenvectors for $T$ or they are 0. In other words, $\mathcal{V}$ is spanned by eigenvectors of $T$ and $T$ is diagonalizable. ■

**Lemma 6** *Suppose $u$ is a generalized eigenvector for $T$ corresponding to the eigenvalue $\lambda$. Then there is a generalized eigenvector $x$ for $A$ corresponding to the eigenvalue $\lambda$ such that $Rx = u$.*

**Proof.** Let $u$ be a (non-zero) generalized eigenvector for $T$ corresponding to the eigenvalue $\lambda$. Since $R$ maps $\mathbf{C}^n$ onto $\mathcal{V}$, there is $y$ in $\mathbf{C}^n$ so that $Ry = u$. If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the distinct eigenvalues of $A$, the Jordan Canonical

Form Theorem applied to $A$ implies that $y = z_1 + z_2 + \cdots + z_k$ where $z_j$ is a generalized eigenvector for $A$ corresponding to the eigenvalue $\lambda_j$. Now, according to Lemma 4, $Rz_j$ is either zero or a generalized eigenvector for $T$ corresponding to the eigenvalue $\lambda_j$. Since

$$u = Ry = Rz_1 + Rz_2 + \cdots + Rz_k$$

and the sum of (non-zero) generalized eigenvectors corresponding to distinct eigenvalues is not a generalized eigenvector at all, we see that $\lambda = \lambda_{j_0}$ for some $j_0$ and $Rz_j = 0$ for $j \neq j_0$. Then we can take $x = z_{j_0}$ and $Rx = Ry = u$.

■

We are now ready to prove the main theorem. Corollary 5 and Lemma 6 can be used to give an easy proof of the theorem in case $A$ is diagonalizable. In this case, if $v_1, \cdots, v_k$ is a basis for $\mathcal{V}$ consisting of eigenvectors for $T$, we can find $x_1, \cdots, x_k$ in $\mathbf{C}^n$ that are eigenvectors for $T$ so that $Rx_j = v_j$. Then we can take $M = \operatorname{span}\{x_j\}$ and the restriction of $R$ to $M$ is the required isomorphism.

If $A$ is not diagonalizable, the situation is not so simple. The proof of the theorem will show that the Jordan chain structure for $T$ is the same as part of the Jordan chain structure for $A$ and it is this fact that leads to the isomorphism we seek. However, the following example shows that $R$ does not necessarily implement any isomorphism of the sort in the theorem! It is this complication that makes the proof less than straightforward.

**Example.** Let

$$A = \begin{pmatrix} 6 & 3 & -2 \\ -7 & -3 & 3 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 2 & 0 \\ -4 & 1 \end{pmatrix}$$

Then relative to the spanning set $(2,1)$, $(1,2)$, $(-1,1)$ for $\mathbf{C}^2$, $A$ is a redundant matrix for $T$: letting $R$ be the matrix with these columns, we see

$$RA = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 6 & 3 & -2 \\ -7 & -3 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & -2 \\ -7 & -2 & 5 \end{pmatrix}$$

and also

$$TR = \begin{pmatrix} 2 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & -2 \\ -7 & -2 & 5 \end{pmatrix}$$

The transformation $T$ is diagonalizable with eigenvectors $(0, 1)$ and $(1, -4)$ corresponding to the eigenvalues 1 and 2 respectively. However, the redundant matrix $A$ is not diagonalizable. $x_1 = (1, -1, 1)$ and $x_2 = (-2, 3, -1)$ are generalized eigenvectors for $A$ corresponding to the eigenvalue 1 such that $(A - I)x_2 = x_1$ and $(A - I)x_1 = 0$ and $y_1 = (1, -2, -1)$ is an eigenvector for $A$ corresponding to the eigenvalue 2.

Moreover, the only two-dimensional invariant subspaces for $A$ are $M_1 = \mathrm{span}\{x_1, x_2\}$ and $M_2 = \mathrm{span}\{x_1, y_1\}$ and these are the only candidates for the invariant subspace $M$ of Theorem 3. However, both $R(M_1)$ and $R(M_2)$ are one-dimensional because $Rx_1 = 0$! This means that there can be no invariant subspace for $A$ on which $R$ implements an isomorphism with $T$! The proof below shows that the restriction of $A$ to $M_2$ is isomorphic to $T$ on $\mathbf{C}^2$. The complication of the proof is due to the fact that we must use $R$ to produce the isomorphism, yet $R$ does not itself implement it. ■

**Proof.** (of Theorem 3.) The Jordan Canonical Form Theorem shows that $\mathbf{C}^n$ is a direct sum of the subspaces of generalized eigenvectors for $A$ and that $\mathcal{V}$ is a direct sum of the subspaces of generalized eigenvectors for $T$. Lemmas 6 shows that if $\lambda$ is an eigenvalue of $T$, then it is also an eigenvalue of $A$. Moreover, if $N_\lambda$ and $M_\lambda$ are the subspaces of generalized eigenvectors corresponding to the eigenvalue $\lambda$ for $T$ and $A$ respectively, Lemmas 4 and 6 show that $R$ maps $M_\lambda$ onto $N_\lambda$.

Let

$$\{u_{i,j} : i = 1, 2, \cdots, \ell_j \quad \text{and} \quad j = 1, 2, \cdots, p\}$$

be a basis for $N_\lambda$ such that $(T - \lambda I)u_{i,j} = u_{i-1,j}$ for $i > 1$ and $(T - \lambda I)u_{1,j} = 0$ for $j = 1, 2, \cdots, p$ and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_p \geq 1$. Similarly, let

$$\{v_{i,j} : i = 1, 2, \cdots, k_j \quad \text{and} \quad j = 1, 2, \cdots, r\}$$

be a basis for $M_\lambda$ such that $(A - \lambda I)v_{i,j} = v_{i-1,j}$ for $i > 1$ and $(A - \lambda I)v_{1,j} = 0$ for $j = 1, 2, \cdots, r$ and $k_1 \geq k_2 \geq \cdots \geq k_r \geq 1$.

We want to show $r \geq p$ and $k_1 \geq \ell_1$, $k_2 \geq \ell_2$, $\cdots$, $k_p \geq \ell_p$.

If $p_1 = p$, and $p_2$ satisfies $\ell_{p_2} \geq 2$ and $\ell_{p_2+1} \leq 1$, and $p_3$ satisfies $\ell_{p_3} \geq 3$ and $\ell_{p_3+1} \leq 2$, etc., then $p_j$ is the number of Jordan chains for $T$ corresponding to $\lambda$ that have length at least $j$. Now,

$$p_1 = \dim N_\lambda - \dim(T - \lambda I)N_\lambda$$

$$p_2 = \dim(T - \lambda I)N_\lambda - \dim(T - \lambda I)^2 N_\lambda$$

and so on.

Similarly, if $r_1 = r$, and $r_2$ satisfies $k_{r_2} \geq 2$ and $k_{r_2+1} \leq 1$, and $r_3$ satisfies $k_{r_3} \geq 3$ and $k_{r_3+1} \leq 2$, etc., then $r_j$ is the number of Jordan chains for $A$ corresponding to $\lambda$ that have length at least $j$ and

$$r_1 = \dim M_\lambda - \dim(A - \lambda I)M_\lambda$$

$$r_2 = \dim(A - \lambda I)M_\lambda - \dim(A - \lambda I)^2 M_\lambda$$

and so on.

Now for each non-negative integer $j$, $R$ maps $(T - \lambda I)^j N_\lambda$ onto $(A - \lambda I)^j M_\lambda$. Indeed, if $w$ is in $(T - \lambda I)^j N_\lambda$, then there is $u$ in $N_\lambda$ so that $w = (T - \lambda I)^j u$. Lemma 6 says that there is $x$ in $M_\lambda$ so that $Rx = u$. Now $(A - \lambda I)^j x$ is in $(A - \lambda I)^j M_\lambda$ and

$$R(A - \lambda I)^j x = (T - \lambda I)^j Rx = (T - \lambda I)^j u = w$$

so $w$ is in the image of $(A - \lambda I)^j M_\lambda$ under $R$. Conversely, if $z$ is in $(A - \lambda I)^j M_\lambda$, say $z = (A - \lambda I)^j x$ for some $x$ in $M_\lambda$, then by Lemma 4, $Rx$ is in $N_\lambda$ and

$$Rz = R(A - \lambda I)^j x = (T - \lambda I)^j Rx$$

which is in $(T - \lambda I)^j N_\lambda$.

Since $(A - \lambda I)^j M_\lambda \subset (A - \lambda I)^{j-1} M_\lambda$ and $(T - \lambda I)^j N_\lambda \subset (T - \lambda I)^{j-1} N_\lambda$ and $R$ maps between these spaces, it follows that $R$ induces a map from the quotient $(A - \lambda I)^{j-1} M_\lambda / (A - \lambda I)^j M_\lambda$ onto the quotient $(T - \lambda I)^{j-1} N_\lambda / (T - \lambda I)^j N_\lambda$. In particular, the dimension of $(A - \lambda I)^{j-1} M_\lambda / (A - \lambda I)^j M_\lambda$ is greater than or equal to the dimension of $(T - \lambda I)^{j-1} N_\lambda / (T - \lambda I)^j N_\lambda$. That is,

$$\begin{aligned}
r_j &= \dim(A - \lambda I)^{j-1} M_\lambda - \dim(A - \lambda I)^j M_\lambda \\
&= \dim \left( \frac{(A - \lambda I)^{j-1} M_\lambda}{(A - \lambda I)^j M_\lambda} \right) \\
&\geq \dim \left( \frac{(T - \lambda I)^{j-1} N_\lambda}{(T - \lambda I)^j N_\lambda} \right) \\
&= \dim(T - \lambda I)^{j-1} N_\lambda - \dim(T - \lambda I)^j N_\lambda \\
&= p_j
\end{aligned}$$

Moreover, this holds for $j = 1$ so $r \geq p$.

Since the number of chains of each length determines the sizes of the $\ell_j$ and $k_j$ and since they are ordered by size, we can conclude that $k_j \geq \ell_j$ for each $j$.

The above discussion shows that because $k_j \geq \ell_j$ and $r \geq p$, we can define $M'_\lambda$ by

$$M'_\lambda = \text{span}\{v_{i,j} : i = 1, 2, \cdots, \ell_j \ \text{ and } \ j = 1, 2, \cdots, p\}$$

Clearly $M'_\lambda$ is invariant for $A - \lambda I$, hence for $A$ and we can define $Q$ from $M'_\lambda$ onto $N_\lambda$ as the linear map that satisfies

$$Qv_{i,j} = u_{i,j} \quad \text{for } 1 \leq i \leq \ell_j \ \text{ and } \ 1 \leq j \leq p$$

Now since the $v_{i,j}$ are a basis for $M'_\lambda$ and the $u_{i,j}$ are a basis for $N_\lambda$, $Q$ is invertible on this subspace. Also,

$$\begin{aligned} QAv_{i,j} &= Q(A - \lambda I)v_{i,j} + \lambda Q v_{i,j} = Qv_{i-1,j} + \lambda Q v_{i,j} \\ &= u_{i-1,j} + \lambda u_{i,j} = (T - \lambda)u_{i,j} + \lambda u_{i,j} \\ &= TQv_{i,j} \end{aligned}$$

for $i > 1$ and

$$\begin{aligned} QAv_{1,j} &= Q(A - \lambda I)v_{1,j} + \lambda Q v_{1,j} = \lambda Q v_{1,j} \\ &= \lambda u_{1,j} = (T - \lambda)u_{1,j} + \lambda u_{1,j} = TQv_{1,j} \end{aligned}$$

so $QA = TQ$ on $M'_\lambda$. Letting $M$ be the span, for all the eigenvalues of $T$, of the $M'_\lambda$ and extending $Q$ linearly to all of $M$, we get an invertible map of $M$ onto the span, for all the eigenvalues of $T$, of the $N_\lambda$ which is, by the Jordan Canonical Form Theorem, all of $\mathcal{V}$.

To prove the converse, we essentially reverse the proof above. Suppose $T$ is a linear transformation on $\mathcal{V}$ and that $A$ is an $n \times n$ matrix that has an invariant subspace $M$ such that the restriction of $A$ to $M$ is isomorphic to $T$. The generalized eigenvectors for the restriction of $A$ to $M$ are also generalized eigenvectors for $A$ on $\mathbf{C}^n$, so each eigenvalue of $T$ is also an eigenvalue of $A$ and for each Jordan chain of $T$ there is a corresponding Jordan chain of $A$ that is at least as long. That is, suppose, as in the Jordan Canonical Form Theorem, $T$ has eigenvalues $\lambda_1$, $\lambda_2$, $\cdots$, $\lambda_m$, not necessarily distinct, and

$$\{u_{i,j} : 1 \leq i \leq \ell_j \ \text{ and } \ 1 \leq j \leq m\}$$

is a basis for $\mathcal{V}$ where $(T-\lambda_j I)u_{i,j} = u_{i-1,j}$ for $1 < i \leq k_j$ and $(T-\lambda_j I)u_{1,j} = 0$ for $j = 1, \cdots, m$. We can enumerate the eigenvalues of $A$ and choose a basis for $\mathbf{C}^n$

$$\{x_{i,j} : 1 \leq i \leq k_j \quad \text{and} \quad 1 \leq j \leq s\}$$

$(A - \lambda_j I)x_{i,j} = x_{i-1,j}$ for $1 < i \leq k_j$ and $(A - \lambda_j I)x_{1,j} = 0$ for $j = 1, \cdots, s$ where $m \leq s$ and $\ell_j \leq k_j$ for each $j$.

Now let define $R$ on $\mathbf{C}^n$ by $Rx_{i,j} = u_{i-k_j+\ell_j,j}$ for $k_j - \ell_j + 1 \leq i \leq k_j$ and $1 \leq j \leq m$ and $Rx_{i,j} = 0$ otherwise. It is easily checked that $R$ maps $\mathbf{C}^n$ onto $\mathcal{V}$ and that $TR = RA$, so by Theorem 1, $A$ is a redundant matrix for $T$. ∎

## 4 Application

If $\mathcal{V}$ is a vector space with basis $v_1, v_2, \cdots, v_n$, then the *tensor product* $\mathcal{V} \otimes \mathcal{V}$ is the vector space of dimension $n^2$ with basis $v_1 \otimes v_1$, $v_1 \otimes v_2$, $\cdots$, $v_1 \otimes v_n$, $v_2 \otimes v_1$, $\cdots$, $v_n \otimes v_n$. Then, we can assign meaning to expressions like $u \otimes v$ where $u = 3v_1 + 2v_2$ and $v = -5v_1 + 4v_2$ by defining

$$u \otimes v = (3v_1 + 2v_2) \otimes (-5v_1 + 4v_2) = -15v_1 \otimes v_1 + 12v_1 \otimes v_2 - 10v_2 \otimes v_1 + 8v_2 \otimes v_2$$

With these definitions, it is possible to show that $\mathcal{V} \otimes \mathcal{V}$ does not depend on the choice of basis.

Now if $S$ and $T$ are linear transformations on $\mathcal{V}$, we can define $S \otimes T$ on $\mathcal{V} \otimes \mathcal{V}$ by $(S \otimes T)(u \otimes v) = (Su) \otimes (Tv)$ and we can show that this definition makes sense. Since $Su = \lambda u$ and $Tv = \mu v$ implies $(S \otimes T)(u \otimes v) = \lambda\mu u \otimes v$, it easy to see that the eigenvalues of $S \otimes T$ are just all possible products of the eigenvalues of $S$ and $T$.

Moreover, if $A$ and $B$ are the matrices for $S$ and $T$ with respect to the basis $\{v_j\}$, then

$$A \otimes B = \begin{pmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1n} & a_{12}b_{11} & \cdots & a_{12}b_{1n} & \cdots & a_{1n}b_{1n} \\ & \vdots & & & \vdots & & \vdots & \\ a_{11}b_{n1} & \cdots & a_{11}b_{nn} & a_{12}b_{n1} & \cdots & a_{12}b_{nn} & \cdots & a_{1n}b_{nn} \\ a_{21}b_{11} & \cdots & a_{21}b_{1n} & a_{22}b_{11} & \cdots & a_{22}b_{1n} & \cdots & a_{2n}b_{1n} \\ & \vdots & & & \vdots & & \vdots & \\ a_{n1}b_{n1} & \cdots & a_{n1}b_{nn} & a_{n2}b_{n1} & \cdots & a_{n2}b_{nn} & \cdots & a_{nn}b_{nn} \end{pmatrix}$$

is the matrix for $S \otimes T$ with respect to the basis $\{v_i \otimes v_j\}$ so $A \otimes B$ has these eigenvalues as well.

Now we can also define the *symmetric tensor product* $\mathcal{V} \otimes_s \mathcal{V}$ as the vector space space of dimension $n(n+1)/2$ with basis $v_1 \otimes_s v_1$, $v_1 \otimes_s v_2$, $\cdots$, $v_1 \otimes_s v_n$, $v_2 \otimes_s v_2$, $\cdots$, $v_{n-1} \otimes_s v_{n-1}$, $v_{n-1} \otimes_s v_n$, $v_n \otimes_s v_n$. That is, for $u$ and $v$ as above

$$u \otimes_s v = v \otimes_s u = -15 v_1 \otimes_s v_1 + 2 v_1 \otimes_s v_2 + 8 v_2 \otimes_s v_2$$

It is again possible to define $S \otimes_s T$ on $\mathcal{V} \otimes_s \mathcal{V}$ by $(S \otimes_s T)(u \otimes_s v) = Su \otimes_s Tv$. Clearly, all products of eigenvalues of $S$ and $T$ are eigenvalues of $S \otimes_s T$, but it is slightly less clear that no other numbers can be eigenvalues in the case that there are duplicates in the set of products. If $A$ and $B$ are matrices that represent $S$ and $T$ with respect to some basis, then $A \otimes B$ is a redundant matrix for $S \otimes_s T$. This shows that the products of the eigenvalues of $S$ and $T$ are the only eigenvalues of $S \otimes_s T$. This observation may be useful in tackling other problems as well.

# References

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