Hilbert Space Operators That Are Subnormal in the Krein Space Sense

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Motivated by some concrete examples that have been studied in other contexts (Example 3), we generalize the concept of subnormality for operators on Hilbert space to include the possibility that the space on which the normal extension is defined is a Krein space instead of a Hilbert space. In this paper, we develop some of the basic properties of this new class, the Krein subnormal operators. In recently published work [**?**], Wu Jingbo studies a different generalization of subnormality, called J-subnormality, and proves the striking theorem that *every* bounded operator on Hilbert space is J-subnormal. Our definition is much more restrictive; Krein subnormal operators have many properties analogous to subnormal operators.

The most important of these properties, and indeed, the ones that provided the motivation for the definition, are moment conditions. In the study of subnormal operators, moment conditions have provided models from which many of the structural theorems have been proved as well as providing the tools for proofs that specific operators are subnormal. One of our goals is to find analogues to these moment conditions. For example, we show (Theorem 5) that a cyclic Krein subnormal operator has a model that is multiplication by z in a Hilbert space of functions in which the analytic polynomials are dense. We conclude with some questions that indicate areas for further exploration of the analogues between subnormality and Krein subnormality, including questions on possible moment conditions.

A complex vector space $\mathcal{K}$ is called a *Krein space* ([**?**]) if it has an indefinite inner product $<\cdot,\cdot>$ given by

$$<x,y> = [Jx,y]$$

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where $[\cdot,\cdot]$ is a positive definite inner product that makes $\mathcal{K}$ a Hilbert space and J is a bounded operator such that $J = J^{-1} = J^{[*]}$. The operator J will be called a *fundamental symmetry*, $P_+ = \frac{1}{2}(I+J)$ and $P_- = \frac{1}{2}(I-J)$ will be called *fundamental projectors*, and the direct sum decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ where $\mathcal{K}^{\pm} = P_{\pm}\mathcal{K}$ will be called the *fundamental decomposition.* (Here and throughout, ${}^{[*]}$ denotes the Hilbert space adjoint on $\mathcal{K}$, whereas * or ${}^{\langle * \rangle}$ denotes the adjoint with respect to the indefinite inner product. Similarly, we write S is [normal] when $S^{[*]}S = SS^{[*]}$ and normal or J-normal or $\langle$normal$\rangle$ when $S^*S = SS^*$ and so forth.)

Definition We say the operator T_2 on the Krein space $(\mathcal{K}_2, \langle\cdot,\cdot\rangle_2)$ is an *extension* of the operator T_1 on the Krein space $(\mathcal{K}_1, \langle\cdot,\cdot\rangle_1)$ if

(i) $\mathcal{K}_1 \subset \mathcal{K}_2$,

(ii) $\langle x,y\rangle_1 = \langle x,y\rangle_2$ for all x,y in $\mathcal{K}_1$, and

(iii) $T_1(x) = T_2(x)$ for all x in $\mathcal{K}_1$.

Definition ([**?**]) If $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space and A is a bounded operator on $\mathcal{H}$, we say A is *J-subnormal* if there is an extension T of A on the Krein space $(\mathcal{K}, \langle\cdot,\cdot\rangle_1)$ such that T is continuous and J-normal.

In particular, $\mathcal{H}$ is a positive definite subspace of $\mathcal{K}$ and $\mathcal{H}$ is T invariant.

The following easy lemma is preparation for the main definition in the paper. It shows that commuting with the fundamental symmetry or the fundamental projectors can be interpreted as a consistency condition on the adjoints of N or as block diagonalizability.

Lemma 1 *If T is an operator on $\mathcal{K}$, then the following are equivalent:*

(1) $TJ = JT$.

(1′) $TP_+ = P_+T$.

(1″) $TP_- = P_-T$.

(2) $T^{[*]} = T^{\langle * \rangle}$.

(3) $\mathcal{K}^+$ and $\mathcal{K}^-$ are invariant subspaces of T.

Definition Operators on a Krein space $\mathcal{K}$ with fundamental symmetry J that satisfy the equivalent conditions of Lemma 1 will be called *fundamentally reducible* (with respect to J).

Proof.($1 \Leftrightarrow 1' \Leftrightarrow 1''$) Clear.

($1 \Leftrightarrow 2$) For all x and y in $\mathcal{K}$,

$$\langle T^*x, y\rangle = \langle x, Ty\rangle = [Jx, Ty] = [T^{[*]}Jx, y] = [JT^{[*]}x, y] = \langle T^{[*]}x, y\rangle .$$

Since J is invertible, the indefinite inner product is non-degenerate and the conclusion follows.

(1⇒3) We have $T\mathcal{K}^+ = TP_+\mathcal{K} = P_+T\mathcal{K} \subset \mathcal{K}^+$ and $T\mathcal{K}^- = TP_-\mathcal{K} = P_-T\mathcal{K} \subset \mathcal{K}^-$.

(3⇒1) For every x, we have $P_+Tx+P_-Tx = Tx = TP_+x+TP_-x$. By the invariance hypothesis, TP_+x and TP_- are in $\mathcal{K}^+$ and $\mathcal{K}^-$ respectively. By the uniqueness of the decomposition, we have $P_+Tx = TP_+x$ and $P_-Tx = TP_-x$.

□

Definition We say the bounded operator A on $\mathcal{H}$, is *Krein subnormal* if there is a continuous, J-normal, fundamentally reducible, operator N on a Krein space $\mathcal{K}$ that extends A such that $\mathcal{H}$ is a closed subspace of $\mathcal{K}$. We call N a *Krein normal extension* of A.

Definition Let μ be a real regular Borel measure with compact support in the complex plane. By $K^2(|\mu|)$ we mean the Krein space of functions in $L^2(|\mu|)$ with the (indefinite) inner product

$$<f,g> = \int f(z)\overline{g(z)}\, d\mu(z).$$

If $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ and support$(\mu) = E^+ \cup E^-$ is the associated Hahn decomposition, then the canonical fundamental symmetry J is the operator on $K^2(|\mu|)$ of multiplication by $\chi_{E^+} - \chi_{E^-}$. The operator M_μ of multiplication by z is a continuous, fundamentally reducible, J-normal operator on $K^2(|\mu|)$.

Example Let $\mu = \mu^+ - \mu^-$ where $d\mu^+ = d\theta/2\pi$ on the circle $r = 1$ and $d\mu^- = d\theta/8\pi$ on the circle $r = 1/2$ where, as usual, $z = re^{i\theta}$. Let $\mathcal{H}$ be the Hilbert space of $H^2(\partial D)$ functions with the (positive definite) inner product

$$<f,g> = \int f(z)\overline{g(z)}\, d\mu(z).$$

As is easily seen, the set $\{e_n : n \in \mathbf{N}\}$ is an orthonormal basis where

$$e_n = \frac{2^{n+1}}{\sqrt{4^{n+1}-1}} z^n.$$

The operator A of multiplication by z on $\mathcal{H}$ is the weighted shift with the (decreasing) weight sequence

$$w_n = \frac{1}{2}\sqrt{\frac{4^{n+2}-1}{4^{n+1}-1}}.$$

It is easy to check that A is a Krein subnormal operator whose Krein normal extension is M_μ on $K^2(|\mu|)$.

□

The following are motivating examples in that they are interesting operators in their own right, and one is led to ask about the structural differences between the examples that are subnormal and the examples that are Krein subnormal but not subnormal.

Example Cowen and Kreite [**?**] investigated the composition operators C_φ defined on H^2 by $C_\varphi f = f \circ \varphi$ where

$$\varphi(z) = \frac{(r+s)z + (1-s)c}{r(1-s)\overline{c}z + (1+sr)}$$

for some r, s with $-1 < r \leq 1$ and $0 < s < 1$. It is shown that C_φ^* is unitarily equivalent to multiplication by

$$s^{\left(\frac{1+z}{1-z} - \frac{1}{2}\right)}$$

on the Hilbert space $P^2(\mu_r)$, the closure of the analytic polynomials in the Krein space $K^2(\mu_r)$, where μ_r is a real measure supported in the closed unit disk. For $0 \leq r \leq 1$, the measures μ_r are positive and the corresponding operators C_φ^* are subnormal in the usual sense. However for $-1 < r < 0$, the measures μ_r are non-positive, so the extension is defined on a Krein space that is not a Hilbert space, and the operators C_φ^* are Krein subnormal, not subnormal in the usual sense.

□

Krein subnormal operators are special cases of J-subnormal operators that retain some of the special properties associated with subnormal operators on Hilbert space.

Definition If A is Krein subnormal on $\mathcal{H}$ and N is a Krein normal extension on the Krein space $\mathcal{K}$, we say N is a *minimal* Krein normal extension of A if span$\{(N^*)^n\mathcal{H}\}$ is dense in $\mathcal{K}$.

It is not immediately clear that a Krein subnormal operator *has* a minimal Krein normal extension. This will be investigated further below (Theorem 4).

Let A be Krein subnormal on $\mathcal{H}$ with Krein normal extension N on $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$. Then by Lemma 1, $N = N_+ + N_-$ where N_+ and N_- are the restrictions of N to $\mathcal{K}^+$ and $\mathcal{K}^-$. Let $\mathcal{H}^+ = P_+\mathcal{H}$. By Lemma IV.7.1

of [**?**], P_+ is an isomorphism of $\mathcal{H}$ onto $\mathcal{H}^+$; we will abuse the notation and write P_+^{-1} for the inverse of P_+ considered as a map of $\mathcal{H}$ onto $\mathcal{H}^+$. Now if $x = x^+ + x^-$ is in $\mathcal{H}$, then $Ax = Nx = N_+x^+ + N_-x^-$, so $N_+x^+ = P_+(Ax)$ which is in $\mathcal{H}^+$. Thus, $\mathcal{H}^+$ is invariant for the normal operator N_+; denoting the restriction of N_+ to $\mathcal{H}^+$ by A_+ this says that A_+ is subnormal.

Analogously, letting $\mathcal{H}^- = \text{closure}(P_-\mathcal{H})$, we see that $\mathcal{H}^-$ is invariant for N_- and that A_-, the restriction of N_- to $\mathcal{H}^-$, is subnormal. By Theorem V.5.7 of [**?**] the angular operator $\alpha\colon \mathcal{H}^+ \mapsto \mathcal{H}^- \subset N^-$ given by $\alpha(P_+x) = P_-x$ is a strict contraction. It is easily seen from these definitions that $\alpha A_+ = A_-\alpha$.

We will distinguish the linear transformation N on $\mathcal{K}$ regarded as a Krein space from the same linear transformation on $\mathcal{K}$ regarded as a Hilbert space by using $\check{N}$ and $\check{\mathcal{K}}$ for the latter. The subspace $\check{\mathcal{H}}$, that is, $\mathcal{H} \subset \check{\mathcal{K}}$, is closed and we denote by β the identification of $\mathcal{H}$ with $\check{\mathcal{H}}$. Now $\check{\mathcal{H}}$ is invariant for $\check{N}$ and, by Lemma 1, $\check{N}$ is [normal] so $\check{A}$ (A on $\check{\mathcal{H}}$) is subnormal.

Definition If A is Krein subnormal on $\mathcal{H}$ and N is a Krein normal extension on the Krein space $\mathcal{K}$, we say N is a *basic* Krein normal extension of A if $\text{span}\{(N_+^*)^n\mathcal{H}^+\}$ is dense in $\mathcal{K}^+$, and $\text{span}\{(N_-^*)^n\mathcal{H}^-\}$ is dense in $\mathcal{K}^-$.

Clearly a minimal Krein normal extension is a basic Krein normal extension. From the general theory of subnormal operators, basic Krein normal extensions always exist, but basic Krein normal extensions are not as nice as one might expect.

Example *Basic Krein normal extensions of a Krein subnormal operator are not necessarily unique.*

Let S on $\ell^2(\mathbf{N})$ be the usual unilateral shift. Then U on $\ell^2(\mathbf{Z})$, the usual bilateral shift, is a minimal normal extension, so S is in a trivial way a Krein subnormal operator with basic Krein normal extension U. Let s and t be positive numbers such that $1 + t^2 = s^2$, and let $\mathcal{K}$ be the Krein space with orthogonal basis $\{\cdots, e_{-2}, e_{-1}, e_0', e_0, e_1, e_2, \cdots\}$, where $[e_n, e_n] = 1$ for all n, and $[e_0', e_0'] = -1$.

Let W be the weighted unilateral shift on $\text{span}\{e_n : n \geq 0\} \subset \mathcal{K}^+$ defined by $We_0 = s^{-1}e_1$ and $We_n = e_{n+1}$ for $n > 0$. The operator W is subnormal; let $\tilde{W}$ be its minimal normal extension on $\mathcal{K}^+$ and let N be the J-normal operator $N = \tilde{W} \oplus 0$ on $\mathcal{K}$. Define τ by

$$\tau(x) = sx_0e_0 + tx_0e_0' + \sum_{k=1}^{\infty} x_ke_k$$

for $x = (x_0, x_1, \cdots)$ in $\ell^2(\mathbf{N})$. Letting $\mathcal{H}$ be the range of τ, we see that $\mathcal{H}$ is a closed Hilbert subspace of $\mathcal{K}$, that τ is a unitary operator from $\ell^2(\mathbf{N})$ onto $\mathcal{H}$, that $\mathcal{H}$ is invariant for N, and that

$$\tau S = N\tau.$$

From the definition of W and $\mathcal{K}$, we see that $\mathcal{H}^+ = \text{span}\{e_0, e_1, e_2, \cdots\}$ and N^+ is $\tilde{W}$, and that $\mathcal{H}^- = \text{span}\{e_0'\}$ and N^- is 0. Thus, N and U are basic Krein normal extensions of S and since U is defined on a Hilbert space, and N is not, they are clearly not unitarily equivalent.

□

The following theorem records some interesting elementary observations.

Theorem 1 *Let A be a Krein subnormal operator on $\mathcal{H}$ with basic Krein normal extension N on $\mathcal{K}$ and let the notation be as above. The following hold:*

(1) A is similar to the subnormal operator A_+.

(2) A is similar to the subnormal operator $\check{A}$.

(3) For λ complex, $A - \lambda I$ is Krein subnormal and $(A - \lambda I)_\pm = A_\pm - \lambda I$.

(4) $\sigma(A) = \sigma(A_+) = \sigma(\check{A}) \supset \sigma(A_-)$.

(5) $\|A\| \geq \|A_+\| = \|\check{A}\| \geq \|A_-\|$.

Proof. *(1):* $A = P_+^{-1} A_+ P_+$.

(2): For x in $\mathcal{H}$, we have

$$\begin{aligned} \|x\|_{\mathcal{H}}^2 &= \langle x, x\rangle = [Jx, x] = [P_+x, P_+x] - [P_-x, P_-x] \\ &\leq [P_+x, P_+x] + [P_-x, P_-x] = \|x\|_{\check{\mathcal{H}}}^2. \end{aligned}$$

Thus, the map β^{-1} is a contraction. Since both $\check{\mathcal{H}}$ and $\mathcal{H}$ are complete, the open mapping theorem implies that β is bounded. The equality $A = \beta^{-1}\check{A}\beta$ implies A is similar to $\check{A}$.

(3): Clear.

(4): By *(1)* and *(2)*, A, A_+, and $\check{A}$ are similar so their spectra are equal. Moreover A_- is subnormal, the range of P_- is dense in $\mathcal{H}^-$, and $P_-A = A_-P_-$, so [**?**, Theorem 1] implies that A_- is invertible whenever A is, which by *(3)*, proves the final containment.

(5): The spectral radius is no more than the norm for any operator and equality holds for (Hilbert space) subnormal operators, so *(4)* implies *(5)*.

□

The weighted shift of Example 2 show that the inequalities in *(4)* and *(5)* are the best possible in general.

Corollary 2 *If A is a Krein subnormal operator whose spectrum has area zero, then A is normal. In particular, if A is compact and Krein subnormal, then A is normal.*

Proof. By part *(4)* of Theorem 1, $\check{A}$ is a subnormal operator with spectrum having measure zero. By Putnam's Theorem [**?**], $\check{A}$ is normal and $\mathcal{H}$ is a reducing subspace of $\check{N}$. But by Lemma 1, $N^{<*>} = N^{[*]}$, so $\mathcal{H}$ is also a reducing subspace of N and A is normal.

□

Corollary 3 *If A is a Krein subnormal operator and λ is an eigenvalue of A, then ker$(A - \lambda I)$ is a reducing subspace of A on which A is normal.*

Proof. Let $\ker(A - \lambda I) = \mathcal{H}_0$. The restriction of A to $\mathcal{H}_0$ is a Krein subnormal operator with spectrum $\{\lambda\}$, so by Corollary 2, A on $\mathcal{H}_0$ is normal. This implies that each vector in $\mathcal{H}_0$ is also an eigenvector of A^*, so $\mathcal{H}_0$ is invariant for A also, and $\mathcal{H}_0$ is reducing.

□

Of course, many results about subnormal operators concern properties that are invariant under similarity; these remain true of Krein subnormal operators. Among these, the most interesting are the theorems concerning existence of hyperinvariant subspaces.

Since many non-normal operators on a finite dimensional space are similar to normal operators, Krein subnormal operators are not just those similar to subnormal operators.

Theorem 4 *An operator A on $\mathcal{H}$ is Krein subnormal if and only if there is an equivalent inner product $[\cdot,\cdot]'$ on $\mathcal{H}$ such that for any vectors $v_0, \cdots, v_n$ in $\mathcal{H}$,*

$$\left| \sum_{j,k=0}^{n} <A^k v_j, A^j v_k> \right| \leq \sum_{j,k=0}^{n} [A^k v_j, A^j v_k]'.$$

Moreover, every Krein subnormal operator has a minimal Krein normal extension.

Proof. We will show that for Krein subnormal operators, the Hilbert norm on the Krein space satisfies the inequality, and we will show that if the inequality holds then the operator A has a minimal Krein normal extension. ($\Rightarrow$) Let A be a Krein subnormal operator, N its Krein normal extension on the Krein space $\mathcal{K}$, and let J be the fundamental symmetry for $\mathcal{K}$. We see that since J is a self-adjoint contraction that commutes with N

$$\left|\sum_{j,k=0}^{n} <A^k v_j, A^j v_k>\right| = \left|\sum_{j,k=0}^{n}[JA^k v_j, A^j v_k]\right| = \left|\sum_{j,k=0}^{n}[JN^k v_j, N^j v_k]\right|$$

$$= \left|\sum_{j,k=0}^{n}[N^k J v_j, N^j v_k]\right| = \left|\sum_{j,k=0}^{n}[(N^*)^j J v_j, (N^*)^k v_k]\right|$$

$$= \left|[J\sum_{j=0}^{n}(N^*)^j v_j, \sum_{k=0}^{n}(N^*)^k v_k]\right| \leq [\sum_{j=0}^{n}(N^*)^j v_j, \sum_{k=0}^{n}(N^*)^k v_k]$$

$$= \sum_{j,k=0}^{n}[A^k v_j, A^j v_k].$$

The proof of part *(2)* of Theorem 1 shows that the inner products are equivalent on $\mathcal{H}$.
($\Leftarrow$) The operator A is subnormal on $\mathcal{H}$ with respect to the inner product $[\cdot,\cdot]'$ by the Bram-Halmos condition [**?**, page 117], so there is a Hilbert space $\mathcal{K}'$ with inner product $[\cdot,\cdot]'$ and a normal operator N on $\mathcal{K}'$ such that $A = N|_{\mathcal{H}}$ and $\mathcal{K}_0 = \text{span}\{(N^*)^k\mathcal{H} : k \geq 0\}$ is dense in $\mathcal{K}'$.

Letting $\tilde{x} = \sum_{j=0}^{n}(N^*)^j x_j$ in $\mathcal{K}_0$, we see that f defined by

$$f(\tilde{x},\tilde{y}) = f(\sum_{j=0}^{n}(N^*)^j x_j, \sum_{k=0}^{n}(N^*)^k y_k) = \sum_{j,k=0}^{n} <A^k x_j, A^j y_k>$$

is a bilinear Hermitian form and

$$|f(\tilde{x},\tilde{x})| \leq [\tilde{x},\tilde{x}]'.$$

This inequality implies that there is a self-adjoint operator J' on $\mathcal{K}'$ such that $\|J'\| \leq 1$ and

$$f(\tilde{x},\tilde{y}) = [J'\tilde{x},\tilde{y}]'.$$

Define a new non-negative inner product by

$$[\tilde{x},\tilde{y}] = [|J'|\tilde{x},\tilde{y}]'.$$

Complete $\mathcal{K}_0$ in the inner product $[\cdot,\cdot]$ to get $\mathcal{K}$ and extend

$$<\tilde{x},\tilde{y}> = f(\tilde{x},\tilde{y}) = [J'\tilde{x},\tilde{y}]'$$

to get an indefinite inner product making $\mathcal{K}$ into a Krein space.

Now $\mathcal{H}$ is naturally identified as a closed subspace of $\mathcal{K}$. Indeed, since $|J'|$ is a positive contraction and $<\cdot,\cdot>$ and $[\cdot,\cdot]'$ are equivalent, there is a constant c so that for x in $\mathcal{H}$,

$$[x,x] = [|J'|x,x]' \leq [x,x]' \leq c<x,x>.$$

On the other hand,

$$<x,x> = [J'x,x]' \leq [|J'|x,x]' = [x,x].$$

Thus, $<\cdot,\cdot>$ and $[\cdot,\cdot]$ are equivalent, which implies $\mathcal{H}$ is closed in $\mathcal{K}$.

Since $\mathcal{K}_0$ is dense in $\mathcal{K}'$ and

$$[J'N\tilde{x},\tilde{y}]' = f(N\tilde{x},\tilde{y}) = f(\sum_{j=0}^{n}(N^*)^j Nx_j, \sum_{k=0}^{n}(N^*)^k y_k)$$

$$= \sum_{j,k=0}^{n} <A^k Ax_j, A^j y_k> = f(\sum_{j=0}^{n}(N^*)^j x_j, \sum_{k=0}^{n}(N^*)^{k+1} y_k)$$
$$= f(\tilde{x},N^*\tilde{y}) = [J'\tilde{x},N^*\tilde{y}]' = [NJ'\tilde{x},\tilde{y}]',$$

we see that N and J' commute on $\mathcal{K}'$. Since J' is self-adjoint, N also commutes with $|J'|$ and $|J'|^{1/2}$. It follows that N is continuous on $\mathcal{K}$ since for u in $\mathcal{K}'$

$$[Nu,Nu] = [|J'|Nu,Nu]' = [N|J'|^{1/2}u, N|J'|^{1/2}u]'$$
$$\leq \|N\|^2[|J'|u,u]' = \|N\|^2[u,u].$$

Since N commutes with the spectral projections for J', we see that N is fundamentally reducible on $\mathcal{K}$. In addition, it follows from the construction of $\mathcal{K}$ that $\mathcal{K}_0$ is dense in $\mathcal{K}$ so that N is a minimal Krein normal extension of A.

□

The following theorem says that for cyclic Krein subnormal operators, the weighted shift in Example 2 is typical; namely, all such operators are unitarily equivalent to multiplication by z on a positive subspace of $K^2(|\mu|)$ for a real measure μ.

Theorem 5 *Let A be a Krein subnormal operator on $\mathcal{H}$ with cyclic vector y. Then there is a unique real regular Borel measure μ with support contained in $\sigma(A)$ such that*

(1) the operator U defined for analytic polynomials p by

$$U(p(A)y) = p(z)$$

has a unique extension to a unitary operator mapping $\mathcal{H}$ onto a closed subspace of $K^2(|\mu|)$,

(2) the J-normal operator of multiplication by z on $K^2(|\mu|)$ is a minimal Krein normal extension of UAU^{-1} that commutes with the canonical fundamental symmetry of $K^2(|\mu|)$, and

(3) any other minimal, fundamentally reducible, J-normal extension of A is unitarily equivalent to M_μ.

The closed subspace of $K^2(|\mu|)$ mentioned in *(1)* above is clearly the closure of the subspace of analytic polynomials; we will call this subspace $P^2(\mu)$.

Proof. Let A have basic Krein normal extension N on $\mathcal{K}$. Since y is a cyclic vector, $\{p(A)y : p \text{ an analytic polynomial}\}$ is dense in $\mathcal{H}$, and since

$$P_\pm p(A)y = p(A_\pm)y^\pm,$$

$\{p(A_+)y^+\}$ is dense in $\mathcal{H}^+$ and $\{p(A_-)y^-\}$ is dense in $\mathcal{H}^-$. That is, A_+ and A_- are cyclic subnormal operators. By Theorem III.5.3 of [**?**], there are measures $\nu^\pm$ on $\sigma(A_\pm)$ such that $N_\pm$ are unitarily equivalent to multiplication by z on $L^2(\nu^\pm)$. In particular,

$$\begin{aligned}\|p(A)y\|^2 &= \|p(A_+)y^+\|^2 - \|p(A_-)y^-\|^2 \\ &= \int |p(z)|^2\, d\nu^+ - \int |p(z)|^2\, d\nu^-\end{aligned}$$

for all analytic polynomials p. For $\mu = \nu^+ - \nu^-$, let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ and let support$(\mu) = E^+ \cup E^-$ be the associated Hahn decomposition. Then μ is a real regular Borel measure with support contained in $\sigma(A)$. Define the operator U from $\mathcal{H}$ into $K^2(|\mu|)$ by

$$U(p(A)y) = p(z)$$

for analytic polynomials p. Since

$$\|p(A)y\|^2 = \int |p(z)|^2\, d\nu^+ - \int |p(z)|^2\, d\nu^- = \int |p(z)|^2\, d\mu,$$

U has a unique extension to a unitary operator mapping $\mathcal{H}$ onto a subspace of $K^2(|\mu|)$.

Now by Theorem 1 and standard results on the Jordan decomposition, there is a constant c such that for each polynomial p,

$$c\int |p(z)|^2\,d\mu^+ \leq c\int |p(z)|^2\,d\nu^+ \leq \|p(A)y\|^2 \leq \int |p(z)|^2\,d\mu^+.$$

Letting Q_+ and Q_- denote the canonical projectors on $K^2(|\mu|)$, this inequality implies that Q_+ maps $U(\mathcal{H})$ isomorphically onto a subspace of $K^2(|\mu|)^+$. It follows, because Q_+U is an isomorphism defined on the Hilbert space $\mathcal{H}$, that $Q_+U(\mathcal{H})$ is complete and closed, which, in turn, means that $U(\mathcal{H})$ is closed.

Let M denote multiplication by z on $L^2(|\mu|)$, and $Q_\pm$ denote the fundamental projectors on $K^2(|\mu|)$. Since E^+ and E^- are disjoint and the measure μ is regular, for each x in $\mathcal{H}$ and $\epsilon > 0$, there is a polynomial q in two variables such that $\|Q_+(Ux) - q(M, M^*)(Ux)\| < \epsilon$ in $L^2(|\mu|)$ and similarly for Q_-. Therefore,

$$\{(M^*)^n(Ux) : x \in \mathcal{H} \text{ and } n \geq 0\}$$

is dense in $K^2(|\mu|)$ and M is a minimal Krein normal extension of UAU^{-1}.

If $\tilde{\mu}$ is another such measure, then for every polynomial $q(r,s) = \sum a_{i,j} r^i s^j$ in two variables,

$$\int q(z,\overline{z})\,d\mu = <q(M,M^*),1> = \sum a_{i,j} <M^i 1, M^j 1>$$

$$= \sum a_{i,j} <A^i y, A^j y> = \int q(z,\overline{z})\,d\tilde{\mu}.$$

By the Riesz representation theorem, μ and $\tilde{\mu}$ are equal.

□

Corollary 6 *The minimal Krein normal extension of a cyclic Krein subnormal operator is unique up to unitary equivalence.*

Proof. The spectral theorem for cyclic normal operators on Hilbert space implies that a fundamentally reducible J-normal operator is unitarily equivalent to M_μ for some real measure μ. The theorem above shows that the measure is unique.

□

The following corollary is the generalization to Krein subnormality of the moment condition for shifts due to Berger and independently to Gellar and Wallen [**?**, pages 159,160].

Corollary 7 *Let W be the weighted shift $Ae_n = w_n e_{n+1}$ where $w_n > 0$ for all n and $\{e_0, e_1, \cdots\}$ is an orthonormal basis for $\mathcal{H}$ and let ρ denote the spectral radius of A. Then W is Krein subnormal if and only if there is a real regular Borel measure ν on $0 \le r \le \rho$ and a positive number c such that*

$$|w_0 w_1 \cdots w_{n-1}|^2 = \int r^{2n}\, d\nu(r) \ge c \int r^{2n}\, d|\nu|(r).$$

Proof.($\Rightarrow$) Since e_0 is a cyclic vector for W, Theorem 5 implies there is a measure μ supported on the spectrum of W and an isometry U of $\mathcal{H}$ onto $P^2(\mu)$ such that $Ue_0 = 1$ and $UWU^{-1} = M_z$ the operator of multiplication by z. For each real number θ, the operator $Ve_n = e^{in\theta} e_n$ is a unitary operator that implements a unitary equivalence between W and $e^{i\theta}W$. By Theorem 5, the measure μ and its rotation $\mu \circ (e^{-i\theta})$, which is the measure associated with $e^{i\theta} M_z$, must be the same. Defining ν on $0 \le r \le \rho$ by

$$\nu(\Delta) = \mu(\{z \colon |z| \in \Delta\})$$

this means $d\mu = d\nu\, d\theta / 2\pi$. Computing, we find

$$|w_0 w_1 \cdots w_{n-1}|^2 = \|W^n e_0\|^2 = \int |z|^{2n}\, d\mu(z) = \int r^{2n}\, d\nu(r).$$

Since W is Krein subnormal, W and $\check{W}$ are similar and the norms induced by μ and $|\mu|$ are equivalent. In particular, there is a positive number c so that

$$\int |z|^{2n}\, d\mu \ge c \int |z|^{2n}\, d|\mu|$$

which implies

$$\int |r|^{2n}\, d\nu \ge c \int |r|^{2n}\, d|\nu|.$$

($\Leftarrow$) Define μ on $\sigma(W)$ by $d\mu = d\nu\, d\theta/2\pi$. Let $f_n = (w_0 w_1 \cdots w_{n-1})^{-1} z^n$. Then f_n is an orthonormal system and $Ue_n = f_n$ is a unitary operator intertwining W and M_z. To finish the proof we need only show that $U\mathcal{H}$ is closed in $K^2(|\mu|)$, that is, we need to show that the norms defined by μ and $|\mu|$ are equivalent. If p is the analytic polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$,

$$\int |p(z)|^2\, d|\mu| \ge \int |p(z)|^2\, d\mu = \sum |a_j|^2 \int r^{2j}\, d\nu$$

$$\geq c \sum |a_j|^2 \int r^{2j}\, d|\nu| = c \int |p(z)|^2\, d|\mu|$$

so the conclusion follows.

□

Converses of Theorem 5 are more subtle. The most obvious difficulty is that the space corresponding to $P^2(|\mu|)$ may not be positive. The extra condition in Corollary 7 that guarantees closure of $P^2(|\mu|)$ in $K^2(|\mu|)$ is necessary for a converse, and may be difficult to deal with in practice. The following example illustrates the possible failure of this condition.

Example We construct a measure $\nu = \nu_+ - \nu_-$ on $0 \leq r \leq 1$ and integers $j_0, j_1, j_2, \cdots$ such that if the measure μ is defined by $d\mu = d\nu\, d\theta/2\pi$, then μ induces a positive definite inner product on the analytic polynomials, but

$$\int r^{2j_k}\, d\nu < 4^{-k+1} \qquad \text{and} \qquad \int r^{2j_k}\, d|\nu| > 2^{-k}$$

for $k = 0, 1, 2, \cdots$. This means that the series $\sum_{k=0}^{\infty} 2^{k/2} z^{j_k}$ converges absolutely in the inner product space defined by μ but diverges in $L^2(|\mu|)$, so that the norm induced by μ on the closure of the analytic polynomials in $K^2(|\mu|)$ does not make $K^2(|\mu|)$ into a Hilbert space. Therefore, the corresponding weighted shift is not Krein subnormal.

If $\frac{1}{2} = \rho_0 < \rho_1 < \rho_2 < \cdots < 1$ and $0 < \rho'_n < \rho_n$, then let

$$\nu_+ = \sum_{n=0}^{\infty} 2^{-n} \delta_{\rho_n}$$

and let

$$\nu_- = \sum_{n=1}^{\infty} 2^{-n} \delta_{\rho'_n}$$

where δ_ρ is point mass at ρ. Then

$$\int r^{2j}\, d\nu = 4^{-j} + \sum_{n=1}^{\infty} 2^{-n} \left((\rho_n)^{2j} - (\rho'_n)^{2j} \right)$$

and

$$\int r^{2j}\, d|\nu| = 4^{-j} + \sum_{n=1}^{\infty} 2^{-n} \left((\rho_n)^{2j} + (\rho'_n)^{2j} \right).$$

It is easily seen that the corresponding μ induces a positive definite inner product on the analytic polynomials.

We inductively choose integers $j_0 < j_1 < j_2 < \cdots$; numbers $\rho_0 < \rho_1 < \rho_2 < \cdots < 1$; and numbers $0 < \alpha_1 < \alpha_2 < \cdots < 1$ so that the choice $\rho'_n = \alpha_n \rho_n$ satisfies the conditions above. Let $j_0 = 1$, $\alpha_0 = 0$, and $\rho_0 = 1/2$. Choose $j_k > j_{k-1}$ (which implies $j_k > k$) so that

$$(\rho_n)^{2j_k} < 4^{-k} \quad \text{for} \quad n = 0, 1, \cdots, k-1$$

choose α_k with $\alpha_{k-1} < \alpha_k < 1$ so that

$$\left(1 - (\alpha_k)^{2j_k}\right) < 2^{-k}$$

and choose ρ_k with $\rho_{k-1} < \rho_k < 1$ so that

$$(\rho_k)^{2j_k}\left(1 + (\alpha_k)^{2j_k}\right) > 1.$$

Now,

$$\int r^{2j_k}\, d\nu = 4^{-j_k} + \sum_{n=1}^{\infty} 2^{-n}(\rho_n)^{2j_k}\left(1 - (\alpha_n)^{2j_k}\right)$$

$$\leq 4^{-k} + \sum_{n=1}^{k-1} 2^{-n}(\rho_n)^{2j_k}\left(1 - (\alpha_n)^{2j_k}\right) + \sum_{n=k}^{\infty} 2^{-n}(\rho_n)^{2j_k}\left(1 - (\alpha_n)^{2j_k}\right)$$

$$\leq 4^{-k} + \sum_{n=1}^{k-1} 2^{-n}4^{-k} + \sum_{n=k}^{\infty} 2^{-n}2^{-k} < 4^{-k} + 4^{-k} + 2 \cdot 4^{-k} = 4 \cdot 4^{-k}.$$

Moreover,

$$\int r^{2j_k}\, d|\nu| = 4^{-j_k} + \sum_{n=1}^{\infty} 2^{-n}(\rho_n)^{2j_k}\left(1 + (\alpha_n)^{2j_k}\right)$$

$$\geq 2^{-k}(\rho_k)^{2j_k}\left(1 + (\alpha_k)^{2j_k}\right) \geq 2^{-k}.$$

□

Theorem 5 and Corollary 7 give moment conditions applicable in special cases that are analogous to those for subnormal operators and seem likely to be as useful in studying Krein subnormal operators. The following theorem, which is an application of Theorem 4, is a more general operator moment condition analogous to the operator moment condition given by Embry [**?**] for subnormal operators.

Theorem 8 *Let A be a bounded operator on the Hilbert space $\mathcal{H}$. Then A is a Krein subnormal operator if and only if there is a bounded self-adjoint operator measure $S(\cdot)$ supported in the spectrum of A with decomposition $S(\cdot) = S^+(\cdot) - S^-(\cdot)$ such that $S^+(\cdot)$ and $S^-(\cdot)$ are bounded positive operator measures for which*

$$(A^*)^n A^m = \int \overline{z}^n z^m \, S(dz)$$

and for all vectors x and y in $\mathcal{H}$,

$$\int \langle S^{\pm}(dz) A^m x, A^n y\rangle = \int \overline{z}^n z^m \, \langle S^{\pm}(dz)x, y\rangle.$$

Proof.($\Rightarrow$) Let N be a Krein normal extension of A to the Krein space $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ and let P be the [orthogonal] projection of $\mathcal{K}$ onto $\mathcal{H}$. Since N is a normal operator on the Hilbert space $\mathcal{K}$ with the inner product $[\cdot,\cdot]$, the spectral theorem implies that there is a projection valued measure E so that $N = \int z \, E(dz)$, and since N is fundamentally reducible, $N_{\pm} = \int z \, P_{\pm} E P_{\pm}(dz)$. Thus, for x and y in $\mathcal{H}$,

$$\langle (A^*)^n A^m x, y\rangle = \langle A^m x, A^n y\rangle = \langle N^m x, N^n y\rangle$$

$$= \langle N^m x^+, N^n y^+\rangle + \langle N^m x^-, N^n y^-\rangle = [N^m x^+, N^n y^+] - [N^m x^-, N^n y^-]$$

$$= \int \overline{z}^n z^m \, [P_+ E P_+(dz)x, y] - \int \overline{z}^n z^m \, [P_- E P_-(dz)x, y]$$

$$= \int \overline{z}^n z^m \, \langle P_+ E P_+(dz)x, y\rangle - \int \overline{z}^n z^m \, \langle -P_- E P_-(dz)x, y\rangle$$

$$= \int \overline{z}^n z^m \, \langle S(dz)x, y\rangle$$

where

$$S^+(\Delta) = P P_+ E(\Delta) P_+ P, \qquad S^-(\Delta) = -P P_- E(\Delta) P_- P$$

and $S(\Delta) = S^+(\Delta) - S^-(\Delta)$. It is easy to check that the measures satisfy the conditions of the theorem.

($\Leftarrow$) If the bounded positive operator measures S^+ and S^- satisfy the conditions of the theorem, let $\check{S} = S^+ + S^-$ and let

$$[x, y] = \int \langle \check{S}(dz)x, y\rangle.$$

Then

$$\langle x,x\rangle = \int \langle S(dz)x,x\rangle \leq \int \langle \check{S}(dz)x,y\rangle = [x,y]$$

and

$$[x,x] = \int \langle \check{S}(dz)x,x\rangle \leq \|\check{S}(\sigma(A))\| \langle x,x\rangle.$$

This means that $[\cdot,\cdot]$ is equivalent to $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$.

For $x_0, x_1, \cdots, x_n$ in $\mathcal{H}$,

$$\sum_{i,j=1}^{n} \langle A^i x_j, A^j x_i\rangle = \sum_{i,j=1}^{n} \int z^i \overline{z}^j \langle S(dz)x_j, x_i\rangle$$

and

$$\sum_{i,j=1}^{n} [A^i x_j, A^j x_i] = \sum_{i,j=1}^{n} \int z^i \overline{z}^j \langle \check{S}(dz)x_j, x_i\rangle$$

which implies

$$\sum_{i,j=1}^{n} [A^i x_j, A^j x_i] - \sum_{i,j=1}^{n} \langle A^i x_j, A^j x_i\rangle = 2\sum_{i,j=1}^{n} \int z^i \overline{z}^j \langle S^-(dz)x_j, x_i\rangle.$$

Since S^- is a bounded positive operator measure, the right hand side is non-negative. Indeed, if $\sum_{k=1}^{m} \alpha_k \chi_{\Delta_k}$ is a simple function approximating z on $\sigma(A)$, then the integral on the right is approximately

$$\sum_{i,j=1}^{n} \left(\sum_{k=1}^{m} \alpha_k^i \overline{\alpha_k}^j \langle S^-(\Delta_k)x_j, x_i\rangle \right)$$

$$= \sum_{k=1}^{m} \left(\sum_{i,j=1}^{n} \langle S^-(\Delta_k)^{1/2} \overline{\alpha_k}^j x_j, S^-(\Delta_k)^{1/2} \overline{\alpha_k}^i x_i\rangle \right)$$

$$= \sum_{k=1}^{m} \left\| \sum_{j=1}^{n} S^-(\Delta_k)^{1/2} \overline{\alpha_k}^j x_j \right\|^2 \geq 0.$$

Thus,

$$\sum_{i,j=1}^{n} \langle A^i x_j, A^j x_i\rangle \leq \sum_{i,j=1}^{n} [A^i x_j, A^j x_i].$$

In the same way,

$$\sum_{i,j=1}^{n} [A^i x_j, A^j x_i] + \sum_{i,j=1}^{n} \langle A^i x_j, A^j x_i\rangle = 2\sum_{i,j=1}^{n} \int z^i \overline{z}^j \langle S^+(dz)x_j, x_i\rangle,$$

so

$$-\sum_{i,j=1}^{n}\langle A^i x_j, A^j x_i\rangle \le \sum_{i,j=1}^{n}[A^i x_j, A^j x_i].$$

By Theorem 4, this means that A is Krein subnormal.

□

It is illuminating to contrast this result with the result of Corollary 7. The closure of the original space in the Krein space extension, in the shift case, results from the constant c that gives the equivalence of the norms determined by ν and $|\nu|$, whereas, in the operator moment theorem, it results from the assumption that the measures are bounded *in the original norm.*

We conclude with some questions that seem pertinent.

Question 1. *Is the minimal Krein normal extension of every Krein subnormal operator unique?*

We have proved this only in the cyclic case (Theorem 5). In that case, the result is a consequence of the Hahn and Jordan decomposition theorems for real measures. It would seem, in analogy with the cyclic case, that if self-adjoint operator versions of the decomposition theorems could be proved, then the uniqueness would follow as before using Theorem 8.

The corresponding question is unresolved in Wu's work [**?**, **?**, **?**] also. Our conjecture is that minimal normal extensions are unique for Krein subnormal operators but not for J-subnormal operators.

Question 2. *Which operators that are similar to a subnormal operator are actually Krein subnormal operators?*

We have seen that every Krein subnormal operator on a Hilbert space is similar to a subnormal one. It would be interesting to find a condition on the relation between an invertible operator V and a subnormal operator S that would guarantee that $V^{-1}SV$ is a Krein subnormal operator. The finite dimensional case shows that not all operators similar to a subnormal are Krein subnormal, and it would seem likely that the required conditions will be subtle.

Question 3. *What other moment conditions imply that an operator is Krein subnormal? Can the operator moment condition of Theorem 8 be weakened to be a condition on the interval* $0 \le t \le \rho$*?*

A theorem of Lambert [**?**] says that a bounded operator S on a Hilbert space is subnormal if and only if for each vector x there is a positive measure μ_x on $0 \le t \le \rho$ such that $\|A^n x\|^2 = \int t^{2n}\, d\mu_x(t)$. Corollary 7 and the associated example show that one cannot simply replace "positive" by "real" in this theorem. It would be desirable, in general, to be able to get moment conditions on the interval $0 \leq t \leq \rho$ rather than the spectrum of A. This may be impossible because of the cancellation of the positive and negative parts of the mass on the circles that are integrated to give the measure on the interval.

Question 4. *If an operator is both hyponormal and Krein subnormal, is it a subnormal operator?*

It seems likely that the positivity condition of hyponormality and the measure conditions of Krein subnormality would be enough to give the positive measure conditions needed for subnormality.

In addition to these specific questions, there are several general themes that deserve to pursued and further connections sought. For example, McEnnis [**?**] studied fundamental reducibility of J-self-adjoint operators and discovered that some growth conditions on the resolvent were important. Are there similar conditions that apply here? Quasi-normality is one of the important tools in the theory of subnormal operators. Moreover, the proof of Wu's theorem on J- subnormal operators closely resembles proofs concerning quasi-normal operators. Is there an appropriate notion of quasi-normality in this context?