Hyponormality of Toeplitz Operators

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Proc. Amer. Math. Soc. 103(1988) 809-812.

Abstract

For φ in $L^\infty(\partial D)$, let $\varphi = f + \overline{g}$ where f and g are in H^2. In this note, it is shown that the Toeplitz operator T_φ is hyponormal if and only if $g = c + T_{\overline{h}} f$ for some constant c and some function h in $H^\infty(\partial D)$ with $\|h\|_\infty \leq 1$.

For φ in $L^\infty(\partial D)$, the Toeplitz operator T_φ is the operator on H^2 of the unit disk D given by $T_\varphi u = P\varphi u$ where P is the orthogonal projection of $L^2(\partial D)$ onto H^2. An operator A is called hyponormal if its self-commutator $A^*A - AA^*$ is positive. The goal of this paper is to characterize hyponormal Toeplitz operators.

Brown and Halmos began the systematic study of the algebraic properties of Toeplitz operators and showed, [3, page 98], that T_φ is normal if and only if $\varphi = \alpha + \beta\rho$ where α and β are complex numbers and ρ is a real valued function in L^∞. There are many results concerning hyponormality of Toeplitz operators in the literature and properties of hyponormal Toeplitz operators have played an important role in work on Halmos's Problem 5, [7], "Is every subnormal Toeplitz operator either normal or analytic?" but a characterization has been lacking. (For references, see the bibliography; [6] surveys much of the literature.)

[0] *1980 Mathematics Subject Classification (1985 Revision).* Primary 47B35, 47B20.
[0] *Key Words and Phrases.* Toeplitz operator, subnormal operator.
*Supported in part by National Science Foundation Grant DMS 87-10006.

Theorem 1 *If φ is in $L^\infty(\partial D)$, where $\varphi = f + \overline{g}$ for f and g in H^2, then T_φ is hyponormal if and only if*

$$g = c + T_{\overline{h}} f.$$

for some constant c and some function h in $H^\infty(\partial D)$ with $\|h\|_\infty \leq 1$.

The basis of the proof is a dilation theorem; we will use the notation and formulation of Sarason [13, Theorem 1]. The unilateral (forward) shift on H^2 will be denoted by U. Moreover, the proof uses standard results about Hankel operators, for example, see [12]. For ψ in L^∞, the Hankel operator H_ψ is the operator on H^2 given by

$$H_\psi u = J(I - P)(\psi u)$$

where J is the unitary operator from ${H^2}^\perp$ onto H^2

$$J(e^{-in\theta}) = e^{i(n-1)\theta}.$$

Denoting by v^* the function $v^*(e^{i\theta}) = \overline{v(e^{-i\theta})}$, another way to put this is that H_ψ is the operator on H^2 defined by

$$<zuv, \overline{\psi}> \; = \; <H_\psi u, v^*>, \qquad \text{for all } v \in H^\infty. \tag{1}$$

Necessary facts about Hankel operators include

- $H_{\psi_1} = H_{\psi_2}$ if and only if $(I - P)\psi_1 = (I - P)\psi_2$.
- $\|H_\psi\| = \inf\{\|\varphi\|_\infty : (I - P)\psi = (I - P)\varphi\}$.
- $H_\psi^* = H_{\psi^*}$.
- Either H_ψ is one-to-one or $\ker(H_\psi) = \chi H^2$ where χ is an inner function. The closure of the range of H_ψ is H^2 in the former case and $(\chi^* H^2)^\perp$ in the latter.
- $H_\psi U = U^* H_\psi$.

Proof. Let $\varphi = f + \overline{g}$ where f and g are in H^2.

The first step of the proof is one of the equivalences of Proposition 11 of [6]. For every polynomial p in H^2,

$$\begin{aligned}
<(T_\varphi^* T_\varphi - T_\varphi T_\varphi^*)(p), p> &= <T_\varphi p, T_\varphi p> \; - \; <T_\varphi^* p, T_\varphi^* p> \\
&= <fp + P\bar{g}p, fp + P\bar{g}p> \; - \; <P\bar{f}p + gp, P\bar{f}p + gp> \\
&= <\bar{f}p, \bar{f}p> \; - \; <P\bar{f}p, P\bar{f}p> \; - \; <\bar{g}p, \bar{g}p> \; + \; <P\bar{g}p, P\bar{g}p> \\
&= <\bar{f}p, (I-P)\bar{f}p> \; - \; <\bar{g}p, (I-P)\bar{g}p> \\
&= <(I-P)\bar{f}p, (I-P)\bar{f}p> \; - \; <(I-P)\bar{g}p, (I-P)\bar{g}p> \\
&= \|H_{\bar{f}}p\|^2 - \|H_{\bar{g}}p\|^2.
\end{aligned}$$

Since the polynomials are dense in H^2 and since the Hankel and Toeplitz operators involved are bounded, we see that T_φ is hyponormal if and only if for all u in H^2,

$$\|H_{\bar{g}}u\| \leq \|H_{\bar{f}}u\|. \qquad (2)$$

Let K denote the closure of the range of $H_{\bar{f}}$, and let S denote the compression of U to K. Since K is invariant for U^*, the operator S^* is the restriction of U^* to K.

Suppose first that T_φ is hyponormal. Define an operator A on the range of $H_{\bar{f}}$ by

$$A(H_{\bar{f}}u) = H_{\bar{g}}u.$$

If $H_{\bar{f}}u_1 = H_{\bar{f}}u_2$, so that $H_{\bar{f}}(u_1 - u_2) = 0$, then the inequality (2) implies that $H_{\bar{g}}(u_1 - u_2) = 0$ too and it follows that A is well defined. Moreover, inequality (2) implies $\|A\| \leq 1$ so A has an extension to K, which will also be denoted A, with the same norm.

Now by the intertwining formula for Hankel operators and the fact that K is invariant for U^*, we have

$$H_{\bar{g}}U = AH_{\bar{f}}U = AU^*H_{\bar{f}} = AS^*H_{\bar{f}}$$

and also

$$H_{\bar{g}}U = U^*H_{\bar{g}} = U^*AH_{\bar{f}} = S^*AH_{\bar{f}}.$$

Since the range of $H_{\bar{f}}$ is dense in K, we find that $AS^* = S^*A$ on K, or taking adjoints, that

$$SA^* = A^*S.$$

By [13, Theorem 1] (or by the usual theory of the unilateral shift if $K = H^2$), there is a function k in $H^\infty(\partial D)$ with $\|k\|_\infty = \|A^*\| = \|A\|$ such that A^* is the compression to K of T_k. Since K is invariant for $T_k^* = T_{\bar{k}}$, this means that A is the restriction of $T_{\bar{k}}$ to K and

$$H_{\bar{g}} = T_{\bar{k}} H_{\bar{f}}. \tag{3}$$

Conversely, if equation (3) holds for some k in $H^\infty(\partial D)$ with $\|k\|_\infty \leq 1$, then clearly inequality (2) holds for all u, and T_φ is hyponormal.

The proof will be completed by analyzing the relationship given by equation (3). Using the formulation (1), equation (3) holds if and only if for all H^∞ functions u, v,

$$\begin{aligned}
\langle zuv, g\rangle &= \langle H_{\bar{g}} u, v^*\rangle = \langle T_{\bar{k}} H_{\bar{f}} u, v^*\rangle \\
&= \langle H_{\bar{f}} u, kv^*\rangle = \langle zuk^* v, f\rangle \\
&= \langle zuv, \overline{k^*} f\rangle = \langle zuv, T_{\overline{k^*}} f\rangle .
\end{aligned}$$

Since the closed span of $\{zuv : u, v \in H^\infty\}$ is zH^2 this means that equation (3) holds if and only if

$$g = c + T_{\bar{h}} f$$

for $h = k^*$. (Note that $\|k\|_\infty = \|k^*\|_\infty$.)

∎

In the cases for which T_φ is normal, h is a constant of modulus 1 and in the cases for which T_φ is known to be subnormal but not normal, h is a constant of modulus less than 1.

It is of some interest to investigate the uniqueness of the functions h that relate f and g. Suppose h_1 and h_2 are in H^∞ and $c_1 + T_{\overline{h_1}} f = g = c_2 + T_{\overline{h_2}} f$. This is possible if and only if

$$T_{\bar{z}} T_{\overline{h_1}} f = T_{\bar{z}} T_{\overline{h_2}} f,$$

that is, if and only if

$$T_{\overline{zh_1 - zh_2}} f = 0.$$

Thus, f must be in $(z\chi H^2)^\perp$ where χ is the inner factor of $h_1 - h_2$. If f is not in any such subspace, the corresponding function h must be unique for every g. On the other hand, if χ is an inner function such that f is in $(z\chi H^2)^\perp$ and $c_1 + T_{\overline{h_1}} f = g$, then for any h_3 in H^∞ and

$$h_2 = h_1 + z\chi h_3,$$

it follows that $g = c_2 + T_{\overline{h_2}} f$ for some constant c_2.

In [6], the author made the following generalization of the set of g in H^2 for which $T_{f+\overline{g}}$ is hyponormal.

Definition Let $\mathcal{H} = \{v \in H^\infty \ : v(0) = 0 \text{ and } \|v\|_2 \leq 1\}$. For f in H^2, let G_f denote the set of g in H^2 such that for every u in H^2,

$$\sup_{v_0 \in \mathcal{H}} |\langle u v_0, g \rangle| \leq \sup_{v_0 \in \mathcal{H}} |\langle u v_0, f \rangle|$$

To see how this definition is relevant to our work, note that if f is in H^∞ and u is in H^2, then by equation (1),

$$\sup_{v_0 \in \mathcal{H}} |\langle u v_0, f \rangle| = \|H_{\bar{f}} u\|.$$

Thus, when f and g are bounded analytic, $T_{f+\overline{g}}$ is hyponormal if and only if g is in G_f.

For f in H^2, not necessarily the analytic part of a function in L^∞, if we regard $H_{\bar{f}}$ as a bounded operator from H^∞ into H^2, then we may proceed exactly as above to prove the following theorem.

Theorem 2 *If f and g are in H^2, then g is in G_f if and only if*

$$g = c + T_{\overline{h}} f.$$

for some constant c and some function h in $H^\infty(\partial D)$ with $\|h\|_\infty \leq 1$.

We can now easily answer Question 1 of [6].

Corollary 3 *For f in H^2, the following hold.*

(1) f is in G_f.

(2) If g is in G_f, then $g + \lambda$ is in G_f for all complex numbers λ.

(3) G_f is balanced and convex; that is, if g_1 and g_2 are in G_f and $|s_1| + |s_2| \leq 1$, then $s_1 g_1 + s_2 g_2$ is also in G_f.

(4) G_f is weakly closed.

(5) $T_{\overline{\chi}} G_f \subset G_f$ for every inner function χ.

Conversely, if G is a set that satisfies properties (1) to (5), then $G \supset G_f$.

Proof. That G_f has the indicated properties is Theorem 12 of [6].

To prove the converse statement, note that f is in G and by (3), (4), and (5), G contains $T_{\overline{h}} f$ whenever h is in the weakly closed convex hull of the set of inner functions. By a theorem of Marshall [11, Corollary, page 496], the norm closed convex hull of the Blaschke products in H^∞ is the unit ball of H^∞. Property (2) and Theorem 2 now imply the desired inclusion. ∎

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