INEQUALITIES FOR THE ANGULAR DERIVATIVE OF AN ANALYTIC FUNCTION IN THE UNIT DISK

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1. *Introduction*

Let ϕ be a function, analytic in the unit disk, D, that maps the unit disk into itself ($\phi(z) \not\equiv z$). In this paper, we present some inequalities for the angular derivative of ϕ. The more important of these concern the derivative of ϕ at its fixed points in the closed unit disk. Since ϕ and ϕ' need not be continuous in $\bar{D}$ we need to clarify the terms "fixed point" and "derivative of ϕ at a fixed point".

DEFINITION. *If ϕ is analytic in D, $\phi(D) \subset D$, and $|z^*| \leqslant 1$, we say that z^* is a fixed point of ϕ if*

$$\lim_{r \to 1^-} \phi(rz^*) = z^* .$$

If z^* is a fixed point of ϕ, then the derivative of ϕ at z^* is the number

$$\phi'(z^*) = \lim_{r \to 1^-} \phi'(rz^*) .$$

It is a consequence of the theorem of Julia, Carathéodory, and Wolff, [7, p. 306] or [6, p. 57], that if z^* is a fixed point of ϕ, then $\phi'(z^*)$ exists and if $|z^*| = 1$ then $\phi'(z^*)$ is real and $0 < \phi'(z^*) \leqslant \infty$. Schwarz's lemma implies there is at most one fixed point in the open disk, and (unless ϕ is an elliptic Möbius transformation) $0 \leqslant |\phi'(z^*)| < 1$ at such a fixed point. The classical result of Denjoy and Wolff is a generalization of Schwarz's lemma.

DENJOY–WOLFF THEOREM [4, 12]. *If ϕ, not the identity, is analytic in D and if $\phi(D) \subset D$, then ϕ has a unique fixed point a ($|a| \leqslant 1$) for which $|\phi'(a)| \leqslant 1$.*

We shall call this distinguished fixed point the *Denjoy–Wolff point of ϕ*. It is easy to give examples for which the Denjoy–Wolff point is the only fixed point of ϕ, but, as we shall see in Section 2, even for univalent functions the set of fixed points can be quite large.

For general analytic functions, we shall prove the following result. For convenience, we have normalized the function so that its Denjoy–Wolff point is 0 or 1.

THEOREM 4.1. *Let ϕ be analytic in D with $\phi(D) \subset D$; let $z_0, z_1, \ldots, z_n$ be distinct fixed points of ϕ in $\bar{D}$.*

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(i) *If* $z_0 = 0$, *then*

$$\sum_{j=1}^{n} \frac{1}{\phi'(z_j)-1} \leqslant \operatorname{Re}\left(\frac{1+\phi'(0)}{1-\phi'(0)}\right).$$

(ii) *If* $z_0 = 1$ *and* $0 < \phi'(1) < 1$, *then*

$$\sum_{j=1}^{n} \frac{1}{\phi'(z_j)-1} \leqslant \frac{\phi'(1)}{1-\phi'(1)}.$$

(iii) *If* $z_0 = 1$ *and* $\phi'(1) = 1$, *then*

$$\sum_{j=1}^{n} \frac{|1-z_j|^2}{\phi'(z_j)-1} \leqslant 2\operatorname{Re}\left(\frac{1}{\phi(0)}-1\right).$$

Moreover, equality holds if and only if ϕ *is a finite Blaschke product of order* $n+1$ *in case* (i) *or of order* n *in cases* (ii) *and* (iii).

We are not assuming here that $z_0, z_1, \ldots, z_n$ is a complete list of the fixed points of ϕ, but the equality condition fails if we replace n by infinity. This theorem is really a quantitative version of the uniqueness statement in the Denjoy–Wolff theorem. The Denjoy–Wolff theorem says that if z^* is a fixed point of ϕ, not the Denjoy–Wolff point, then $\phi'(z^*) > 1$, whereas Theorem 4.1 tells how much bigger $\phi'(z^*)$ must be.

We use Theorem 4.1 to obtain an inequality relating the angular derivatives at points of $\phi^{-1}(\{\lambda\})$ where $|\lambda| = 1$. (As in the fixed point situation, $\phi(z_j)$ and $\phi'(z_j)$ refer to radial limits and $|\phi(z_j)| = 1$ implies the existence of the derivative.)

THEOREM 8.1. *Let* ϕ *be analytic in* D *with* $\phi(D) \subset D$, *and suppose, for* $j = 1, 2, \ldots, n$, *that* $\phi(z_j) = \lambda$, *where* $|z_j| = 1 = |\lambda|$. *Then*

$$\sum_{j=1}^{n} \frac{1}{|\phi'(z_j)|} \leqslant \operatorname{Re}\frac{\lambda+\phi(0)}{\lambda-\phi(0)}$$

and equality holds if and only if ϕ *is a Blaschke product of order* n.

More can be said when ϕ is univalent, as we see in the following.

THEOREM 6.1. *Let* ϕ *be analytic and univalent in* D *with* $\phi(D) \subset D$ *and suppose the Denjoy–Wolff point* a *of* ϕ *satisfies* $|a| = 1$ *and* $\phi'(a) < 1$. *If* $z_1, z_2, \ldots, z_n$ *are distinct fixed points of* ϕ *on* ∂D, *different from* a, *then*

$$\sum_{j=1}^{n} (\log \phi'(z_j))^{-1} \leqslant -(\log \phi'(a))^{-1}.$$

Moreover, equality holds if and only if

$$\phi(z) = s^{-1}\Big(\sigma^{-1}\big(\sigma(s(z))+1\big)\Big)$$

where

$$s(z) = (a+z)(a-z)^{-1}$$

and

$$\sigma(s) = \sum_{j=1}^{n} \left(\log \phi'(z_j)\right)^{-1} \log\left(s - s(z_j)\right).$$

Theorem 7.1 is a similar result for the case when $|a| < 1$, and Theorem 6.2 is a weaker version treating the case when $\phi'(a) = 1$.

The given condition for equality is equivalent to a qualitative one (closely related to the geometric motivation for the inequality) based on semigroups of iterates. For a positive integer k, we will denote by ϕ_k the k-th iterate of ϕ that is, $\phi_1 = \phi$, $\phi_2 = \phi \circ \phi, \ldots, \phi_k = \phi \circ \phi_{k-1}$. The iterates of ϕ form a discrete semigroup under composition. We shall say that ϕ *can be embedded in a continuous semigroup of iterates* if there is a function $F(z, t)$ defined and continuous for $|z| < 1$ and $t \geqslant 0$, such that $\phi_t(z) = F(z, t)$ is an analytic map of the disk into itself for each t, $F(z, 1) = \phi(z)$, and $\phi_{s+t}(z) = \phi_s(\phi_t(z))$. The equality condition of Theorem 6.1 is equivalent to saying that $D \setminus \phi(D)$ consists of $n-1$ analytic arcs and ϕ can be embedded in a continuous semigroup of iterates $\{\phi_t\}$. As t increases, the arcs comprising $D \setminus \phi_t(D)$ grow analytically toward the Denjoy–Wolff point; the semigroup condition is really a condition on the shape of the omitted arcs.

The proofs of the more difficult results of this paper are based on a Grunsky-type inequality of Nehari [**5**] and Schiffer and Tammi [**10**] (see [**7**, p. 98] and Section 5), but the geometric intuition behind the results is based on a model for iteration of functions analytic in the unit disk developed in [**3**]. In the situation of Theorem 6.1, there is a conformal map σ of the disk into the strip $|\operatorname{Im}\zeta| < W$ such that $\phi(z) = \sigma^{-1}(\sigma(z)+1)$. The domain $\sigma(D)$ nearly extends the width of the strip near $+\infty$ and has (at least) n fingers that extend to $-\infty$:

$$\lim_{r \to 1^-} \sigma(ra) = +\infty \qquad \text{and} \qquad \lim_{r \to 1^-} \sigma(rz_j) = -\infty$$

for $j = 1, 2, \ldots, n$. The Ahlfors distortion theorem can be used to relate the harmonic mean of the widths of the fingers to the derivatives at the fixed points, for example, $\phi'(a) = \exp(-\pi/2W)$.

The inequality of Theorem 6.1 is just the observation that the sum of the widths of the fingers is no more than the width of the strip. The equality condition is that equality arises when the fingers fill the strip and are bounded by rays. In this case, the semigroup is defined by $\phi_t(z) = \sigma^{-1}(\sigma(z)+t)$. In using this method of proof certain technical considerations arise that seem to require special hypotheses, so the proofs given here are non-geometric.

The outline of the paper is as follows. In Section 2, we examine the size of the fixed point set (measure zero in general, capacity zero if ϕ is univalent) and give examples where the fixed point set is large. In Section 3, we prove a geometric version of Theorem 4.1 involving only two fixed points, and in Section 4, we prove the general inequalities. Section 5 is devoted to proving the Grunsky-type inequalities needed for univalent functions. Sections 6 and 7 present the results for univalent functions in which the Denjoy–Wolff point is on the circle (Section 6) and in the disk (Section 7). Finally, Section 8 deals with inequalities for the angular derivative at points of $\phi^{-1}(\{\lambda\})$ where $|\lambda| = 1$.

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2. The fixed point set

By the fixed point set of ϕ, we mean the set

$$F = \left\{ z : |z| \leqslant 1 \quad \text{and} \quad \lim_{r \to 1^-} \phi(rz) = z \right\}.$$

As noted in the introduction, F is non-empty since the Denjoy–Wolff point a is in F. By Schwarz's lemma, either $F \cap D = \emptyset$ or $F \cap D = \{a\}$. In this section, we examine the size of F. Since F is the zero set of the bounded function $\phi(z) - z$, the Lebesgue measure of $F \cap \partial D$ is zero. We can say more if ϕ is univalent.

THEOREM 2.1. *If ϕ (not the identity) is analytic and univalent in D with $\phi(D) \subset D$, then the fixed point set of ϕ has outer capacity zero.*

Proof. We use the following inequality concerning capacity [7, p. 348]. Let f be univalent in D, with $f(D) \subset D$ and $f(0) = 0$. If A and $f(A)$ are subsets of ∂D then $\text{cap}^* A \leqslant |f'(0)|^{1/2} \text{cap}^* f(A)$ (where $\text{cap}^* E$ denotes the outer logarithmic capacity of E).

Since the result is trivial if ϕ is a Möbius transformation, we assume that it is not. Let a be the Denjoy–Wolff point of ϕ, and let F be the fixed point set of ϕ. If $|a| < 1$, we may assume that $a = 0$. Applying the result above to the set $A = F \setminus \{a\}$ we obtain $\text{cap}^* A \leqslant |\phi'(0)|^{1/2} \text{cap}^* A$. Since $|\phi'(0)| < 1$, we conclude that $\text{cap}^* A = 0$, and $\text{cap}^* F = 0$.

For $|a| = 1$, let $\phi_n(0) = b_n$ and $\psi_n(z) = (z - b_n)(1 - \bar{b}_n z)^{-1}$ and $f_n = \psi_n \circ \phi_n$. Then f_n satisfies the hypotheses of the above result for each n, and if K is a compact subset of $\mathbb{C} \setminus \{a\}$ we have $f_n(K \cap F) = \psi_n(K \cap F)$. Thus, by Schwarz's lemma,

$$\text{cap}^*(K \cap F) \leqslant |f_n'(0)|^{1/2} \text{cap}^* f_n(K \cap F) \leqslant \text{cap}^* \psi_n(K \cap F).$$

Now $\lim_{n \to \infty} b_n = a$ and an easy calculation shows that ψ_n converges uniformly on compact subsets of $\mathbb{C} \setminus \{a\}$ to the constant $-a$. This means that $\lim_{n \to \infty} \text{cap}^* \psi_n(K \cap F) = 0$, so that $\text{cap}^*(K \cap F) = 0$. Since this holds for all compact subsets of $\mathbb{C} \setminus \{a\}$, it follows that $\text{cap}^* F = 0$.

The following two examples show that these results are best possible.

EXAMPLE 2.2. Let K be a closed set of measure zero in ∂D. There is a function ϕ analytic in D, continuous in $\bar{D}$ with $\phi(D) \subset D$ whose fixed point set is $K \cup \{0\}$.

Let z_0 be a point of $\partial D \setminus K$. In [9, p. 809] Rudin constructs a function q continuous in $\bar{D}$, analytic in D such that $q(\bar{D})$ is contained in a closed rectangle M with 0 and 1 on the boundary of M and $q^{-1}(\{0\}) = \{z_0\}$ and $q^{-1}(\{1\}) = K$. Let ψ be a conformal map of M onto $\bar{D}$ with $\psi(0) = -1$ and $\psi(1) = 1$. Then $\phi(z) = z\psi(q(z))$ is continuous on $\bar{D}$, analytic in D and ϕ is not the identity since $\phi(z_0) = -z_0$. A point w is in the fixed point set of ϕ if and only if $w = 0$ or $\psi(q(w)) = 1$, that is, if and only if w is in $\{0\} \cup K$.

EXAMPLE 2.3. Let K be a closed set of capacity zero in ∂D and let a be a point of $\partial D \setminus K$. There is a function ϕ analytic and univalent in D with $\phi(D) \subset D$ whose fixed point set is $K \cup \{a\}$.

Let $\zeta(z) = (a+z)(a-z)^{-1}$ and let $E = \zeta(K)$. Then ζ maps the unit disk onto the half plane $H = \{\zeta : \operatorname{Re}\zeta > 0\}$ with $\zeta(a) = \infty$ and E is a compact subset of the imaginary axis with capacity zero. By Evan's theorem [**11**, p. 75], there is a positive measure μ on E such that $\mu(E) = 1$ and $\lim_{x \to 0^+} \int_E \log|x+iy-t|\,d\mu(t) = -\infty$ if and only if iy is in E. Let $h(\zeta) = \int \log(\zeta - t)\,d\mu(t)$ for ζ in H. Since H is convex and since $\operatorname{Re} h'(\zeta) = \int \frac{\operatorname{Re}\zeta}{|\zeta-t|^2}\,d\mu(t) > 0$ for ζ in H, we see that h is univalent. Off E we can continue h across the imaginary axis and since $\operatorname{Re} h' = 0$ on $i\mathbb{R}\setminus E$ the image of this set consists of rays parallel to the real axis. It follows that $h(H) \subset \{w : |\operatorname{Im} w| < \pi/2\}$, and $h(H)$ has the property that $w \in h(H)$ implies that $w + r \in h(H)$ for all $r > 0$. Define ϕ on D by $\phi(z) = \zeta^{-1}(h^{-1}(h(\zeta(z))+1))$. Since $h(\zeta(z)) = \infty$ if and only if $z = a$ and since $h(\zeta(z)) = -\infty$ if and only if z is in K, the fixed point set of ϕ is $K \cup \{a\}$.

3. *Analytic functions: two fixed points*

In this section we examine the case in which only two fixed points are considered. This motivates the more general theorems later and also presents a geometric aspect of the problem which we have been unable to generalize. The choice of fixed points at ± 1 in the theorem is a convenient normalization: conjugation by an appropriate Möbius transformation changes fixed points at a, b on the unit circle to ± 1 and does not change the left side of the inequality. We are grateful to Don Marshall, David Minda and Ken Stephenson for their suggestions in refining the statement and proof of this result.

THEOREM 3.1. *Let ϕ be analytic in D with $\phi(D) \subset D$. If* 1 *and* -1 *are fixed points of ϕ then*

$$\phi'(1)\phi'(-1) \geqslant \sup_{-1<x<1} \left[1 + \frac{4(\operatorname{Im}\phi(x))^2}{(1-|\phi(x)|^2)^2}\right]$$

Proof. From the lemma of Julia, Carathéodory and Wolff [**7**, p. 306] or [**6**, p. 57] we have

$$\phi'(1) = \sup_{z\in D} \left[\frac{|1-\phi(z)|^2}{1-|\phi(z)|^2}\right]\left[\frac{1-|z|^2}{|1-z|^2}\right]$$

and

$$\phi'(-1) = \sup_{z\in D} \left[\frac{|1+\phi(z)|^2}{1-|\phi(z)|^2}\right]\left[\frac{1-|z|^2}{|1+z|^2}\right].$$

Multiplication of these equalities and restriction to the real axis yield the inequality.

This is really a distortion theorem. The expression $4(\operatorname{Im} z)^2[1-|z|^2]^{-2}$ is constant on arcs of circles through $+1$ and -1. Thus, the expression on the right measures how far the image of $(-1, 1)$ deviates from $(-1, 1)$. It will follow from Theorem 4.1 that $\phi'(1)\phi'(-1) > 1$ unless ϕ is a Möbius transformation.

4. *Analytic functions: the general case*

In order to prove the inequalities in the general case, we need the following extension of the Julia–Carathéodory–Wolff lemma.

LEMMA 4.0. *Let g be analytic in D and let* $\operatorname{Re} g(z) > 0$. *Let $z_1, z_2, \ldots, z_n$ be points of the unit circle for which $g(z_j) = \infty$, and let $b_j = \lim_{r\to 1-} \frac{1-r}{1+r} g(rz_j)$. Then*

$$h(z) = g(z) - \sum_{j=1}^{n} b_j \frac{z_j+z}{z_j-z}$$

satisfies $\operatorname{Re} h(z) \geqslant 0$.

Proof. Changing variables in the usual version of the Julia–Carathéodory–Wolff lemma so that it applies to functions analytic in D with positive real part, we get that $h_1(z) = g(z) - b_1 \frac{z_1+z}{z_1-z}$ satisfies $\operatorname{Re} h_1(z) \geqslant 0$. (The proof given in [6, pp. 57–60] is a good starting point.)

Since $\lim_{r\to 1-} \frac{1-r}{1+r} \frac{z_j+rz_k}{z_j-rz_k} = 0$ for $j \neq k$, the conclusion follows by induction.

In the following theorem, we assume that the function has been normalized so that the Denjoy–Wolff point is 0 or 1.

THEOREM 4.1. *Let ϕ be analytic in D, with $\phi(D) \subset D$ and let $z_0, z_1, \ldots, z_n$ be distinct fixed points of ϕ in $\bar{D}$.*

(i) *If $z_0 = 0$ then*

$$\sum_{j=1}^{n} \frac{1}{\phi'(z_j)-1} \leqslant \operatorname{Re} \frac{1+\phi'(0)}{1-\phi'(0)}.$$

(ii) *If $z_0 = 1$ and $0 < \phi'(1) < 1$ then*

$$\sum_{j=1}^{n} \frac{1}{\phi'(z_j)-1} \leqslant \frac{\phi'(1)}{1-\phi'(1)}.$$

(iii) *If $z_0 = 1$ and $\phi'(1) = 1$ then*

$$\sum_{j=1}^{n} \frac{|1-z_j|^2}{\phi'(z_j)-1} \leqslant 2 \operatorname{Re}\left(\frac{1}{\phi(0)} - 1\right).$$

Moreover, equality holds if and only if ϕ is a Blaschke product of order $n+1$ in case (i) *or order n in cases* (ii) *and* (iii).

Proof. (i) Let

$$g(z) = \frac{1+z^{-1}\phi(z)}{1-z^{-1}\phi(z)} = \frac{z+\phi(z)}{z-\phi(z)}.$$

Since $z_0 = 0$, the function $z^{-1}\phi(z)$ is analytic in D and maps D into itself, hence $\operatorname{Re} g(z) > 0$ for z in D. Now

$$\lim_{r\to 1-}\frac{1-r}{1+r}g(rz_j) = \lim_{r\to 1-}\frac{1-r}{1+r}\frac{rz_j+\phi(rz_j)}{rz_j-\phi(rz_j)} = (\phi'(z_j)-1)^{-1}.$$

Thus the lemma shows that

$$h(z) = \frac{1+z^{-1}\phi(z)}{1-z^{-1}\phi(z)} - \sum_{j=1}^{n}(\phi'(z_j)-1)^{-1}\frac{z_j+z}{z_j-z}$$

has positive real part. Setting $z = 0$, we have

$$\operatorname{Re}\frac{1+\phi'(0)}{1-\phi'(0)} - \sum_{j=1}^{n}(\phi'(z_j)-1)^{-1} = \operatorname{Re} h(0) \geqslant 0,$$

which is (i). If equality holds in (i) then the maximum principle implies that $\operatorname{Re} h(z) \equiv 0$ so that $h(z) \equiv i\beta$, β real. It follows that all boundary values of $\dfrac{1+z^{-1}\phi}{1-z^{-1}\phi}$ on ∂D are purely imaginary so that the boundary values of $z^{-1}\phi$ have modulus 1. Since $z^{-1}\phi$ is rational of order n it follows that ϕ is a Blaschke product of order $n+1$. Conversely, if $z^{-1}\phi$ is a Blaschke product of order n, then there are n distinct points $z_1, z_2, \ldots, z_n$ on ∂D at which $z^{-1}\phi$ takes the value 1, that is, at which ϕ has fixed points. Moreover, if $h(z)$ is as above then $\operatorname{Re} h = 0$ on ∂D, so that $\operatorname{Re} h(0) = 0$.

(ii) By the lemma, since $\displaystyle\lim_{r\to 1-}\frac{1-r}{1+r}\frac{1+\phi(r)}{1-\phi(r)} = \phi'(1)^{-1} > 1$, the function $f(z) = \dfrac{1+\phi(z)}{1-\phi(z)} - \dfrac{1+z}{1-z}$ has positive real part. Let $g(z) = f(z)^{-1}$ so that $\operatorname{Re} g(z) > 0$ also. Since $z_1, \ldots, z_n$ are fixed points of ϕ different from 1, we see that $g(z_j) = \infty$ for $j = 1, \ldots, n$. Moreover

$$\lim_{r\to 1-}\frac{1+r}{1-r}\frac{1}{g(rz_j)} = \lim_{r\to 1-}\frac{1+r}{1-r}\left[\frac{1+\phi(rz_j)}{1-\phi(rz_j)} - \frac{1+rz_j}{1-rz_j}\right]$$

$$= \lim_{r\to 1-}\frac{2(1+r)(\phi(rz_j)-rz_j)}{(1-r)(1-\phi(rz_j))(1-rz_j)} = \frac{4(\phi'(z_j)-1)}{|1-z_j|^2}$$

for $j = 1, 2, \ldots, n$. The lemma implies that $h(z) = g(z) - \displaystyle\sum_{j=1}^{n}\frac{|1-z_j|^2}{4(\phi'(z_j)-1)}\frac{z_j+z}{z_j-z}$ has positive real part. By Harnack's inequality $\operatorname{Re} h(z) \geqslant \dfrac{1-|z|}{1+|z|}\operatorname{Re} h(0)$, so that for $0 \leqslant r < 1$, taking real parts and dividing by $1-r$, we obtain

$$(*)\qquad \frac{\operatorname{Re} g(r)}{1-r} - \sum_{j=1}^{n}\frac{|1-z_j|^2}{4(\phi'(z_j)-1)}\frac{1+r}{|z_j-r|^2} \geqslant \frac{1}{1+r}\operatorname{Re} h(0).$$

Now $\lim_{r\to 1^-} \frac{1-r}{g(r)} = \lim_{r\to 1^-} (1-r)\left[\frac{1+\phi(r)}{1-\phi(r)} - \frac{1+r}{1-r}\right] = 2\phi'(1)^{-1}-2$. Taking the limit as $r \to 1^-$ in (*) gives

$$\frac{\phi'(1)}{2(1-\phi'(1))} - \sum_{j=1}^{n} \frac{1}{2(\phi'(z_j)-1)} \geqslant \frac{1}{2}\operatorname{Re} h(0) \geqslant 0$$

which is (ii). The statement about equality follows as before.

(iii) We obtain (*) as in the proof of (ii).
Setting $r = 0$, we get

$$\operatorname{Re} g(0) - \sum_{j=1}^{n} \frac{|1-z_j|^2}{4(\phi'(z_j)-1)} \geqslant \operatorname{Re} h(0) \geqslant 0\,.$$

Since $g(0) = \left(\frac{1+\phi(0)}{1-\phi(0)} - 1\right)^{-1} = \frac{1}{2}\left(\frac{1}{\phi(0)} - 1\right)$, this is (iii).

We note that in (ii) if $n = 1$ and $z_1 = -1$ we get $\frac{1}{\phi'(-1)-1} \leqslant \frac{\phi'(1)}{1-\phi'(1)}$ which is equivalent to $\phi'(1)\phi'(-1) \geqslant 1$. This is the inequality of Theorem 3.1 without the geometric content.

5. *Some Grunsky-type inequalities*

In this section we derive some inequalities needed in later sections. We first give an improvement of Corollary 4.3 of [7, p. 99].

THEOREM 5.1. *Let $f(z) = bz + \ldots$ be univalent in D and have $f(D) \subset D$. If $z_1, z_2 \ldots z_n$ are in D and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are real, then*

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \gamma_j\gamma_k \log\left| b\,\frac{z_j z_k}{f(z_j)f(z_k)}\left(\frac{f(z_j)-f(z_k)}{z_j-z_k}\right)\left(\frac{1-f(z_j)\overline{f(z_k)}}{1-z_j\bar{z}_k}\right)\right|$$
$$\geqslant \left(\log\frac{1}{|b|}\right)^{-1}\left(\sum_{j=1}^{n} \gamma_j \arg\frac{f(z_j)}{bz_j}\right)^2.$$

REMARK. We choose the branch of $\log(f(z)/bz)$ on D that vanishes for $z = 0$. As in [7, p. 98], we define Grunsky-type coefficients $a_{\mu\nu}$ and $a^*_{\mu\nu}$ by

(1) $$\log\frac{f(z)-f(\zeta)}{z-\zeta} = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} a_{\mu\nu} z^{\mu}\zeta^{\nu}$$

and

(2) $$\log\left(1-f(z)\overline{f(\bar{\zeta})}\right) = -\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} a^*_{\mu\nu} z^{\mu}\zeta^{\nu}$$

for z and ζ in D.

Proof. Theorem 4.2 of [7, p. 99] asserts that if $\lambda_1, \lambda_2 \ldots$ are complex numbers then

$$\text{(3)} \qquad \operatorname{Re}\left[\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} a_{\mu\nu}\lambda_\mu\lambda_\nu\right] + \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} a^*_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu \leqslant \sum_{\nu=1}^{\infty}\frac{1}{\nu}|\lambda_\nu|^2$$

when the right-hand series converges.

From (1) above we see that $\operatorname{Re} a_{00} = \log|b|$, that $\log\dfrac{f(z)}{bz} = \sum_{\nu=1}^{\infty} a_{\nu 0}z^\nu$ and that

$$\log\frac{bz\zeta(f(z)-f(\zeta))}{f(z)f(\zeta)(z-\zeta)} = \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} a_{\mu\nu}z^\mu\zeta^\nu.$$

For $\nu = 1, 2, 3, \ldots$ let $\lambda_\nu = i\sum_{j=1}^{n}\gamma_j z_j^\nu$ (which gives the convergence of the series in (3)), and let $\lambda_0 = \left(\log\dfrac{1}{|b|}\right)^{-1}\operatorname{Re}\sum_{\nu=1}^{\infty} a_{\nu 0}\lambda_\nu$. These choices mean that

$$\sum_{\nu=1}^{\infty} a_{\nu 0}\lambda_\nu = i\sum_{j=1}^{n}\gamma_j\sum_{\nu=1}^{\infty} a_{\nu 0}z_j^\nu = i\sum_{j=1}^{n}\gamma_j\log\frac{f(z_j)}{bz_j},$$

that

$$\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} a_{\mu\nu}\lambda_\mu\lambda_\nu = -\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_j\gamma_k\left(\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} a_{\mu\nu}z_j^\mu z_k^\nu\right)$$

$$= -\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_j\gamma_k\log\left(\frac{bz_jz_k(f(z_j)-f(z_k))}{f(z_j)f(z_k)(z_j-z_k)}\right)$$

and that

$$\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} a^*_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu = \sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_j\gamma_k\left(\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} a^*_{\mu\nu}z_j^\mu\bar{z}_k^\nu\right)$$

$$= -\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_j\gamma_k\log\left(1-f(z_j)\overline{f(z_k)}\right).$$

Inserting these in (3) yields

$$\left(\log\frac{1}{|b|}\right)^{-1}\left(\sum_{j=1}^{n}\gamma_j\arg\frac{f(z_j)}{bz_j}\right)^2 - \sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_j\gamma_k\log\left|\frac{bz_jz_k(f(z_j)-f(z_k))(1-f(z_j)\overline{f(z_k)})}{f(z_j)f(z_k)(z_j-z_k)}\right|$$

$$\leqslant \sum_{\nu=1}^{\infty}\frac{1}{\nu}|\lambda_\nu|^2 = \sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_j\gamma_k\log|1-z_j\bar{z}_k|^{-1}$$

which is equivalent to our assertion.

In the following corollary we replace the hypothesis $f(0) = 0$ by the hypothesis $\sum\gamma_j = 0$.

COROLLARY 5.2. *Suppose that $g(z)$ is analytic and univalent in D with $g(D) \subset D$.*

If $z_0, z_1, \ldots, z_n$ are in D and $\gamma_0, \gamma_1, \ldots, \gamma_n$ are real numbers such that $\sum_{j=0}^{n} \gamma_j = 0$ then

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \gamma_j \gamma_k \log \left| \frac{g(z_j)-g(z_k)}{z_j - z_k} \frac{1-g(z_j)\overline{g(z_k)}}{1-\bar{z}_j z_k} \right|$$

$$\geqslant \left(\log \frac{1-|\alpha|^2}{|g'(0)|} \right)^{-1} \left[\sum_{j=0}^{n} \gamma_j \arg \frac{1}{g'(0) z_j} \frac{g(z_j)-\alpha}{1-\bar{\alpha} g(z_j)} \right]^2$$

where $\alpha = g(0)$.

Proof. Let $f(z) = (g(z)-\alpha)(1-\bar{\alpha} g(z))^{-1}$ so that $f(0) = 0$ and $f'(0) = (1-|\alpha|^2)^{-1} g'(0)$. We insert this into the inequality of Theorem 5.1. Since $\sum \gamma_j = 0$, the terms containing b cancel and we obtain

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \gamma_j \gamma_k$$

$$\times \log \left| \frac{z_j z_k (g(z_j)-g(z_k))(1-g(z_j)\overline{g(z_k)})}{f(z_j) f(z_k)(1-\bar{\alpha} g(z_j))(1-\bar{\alpha} g(z_k))(z_j - z_k)(1-z_j \bar{z}_k)(1-\bar{\alpha} g(z_j))(1-\alpha \overline{g(z_k)})} \right|$$

$$\geqslant \left(\log \frac{1-|\alpha|^2}{|g'(0)|} \right)^{-1} \left[\sum_{j=0}^{n} \gamma_j \arg \frac{1}{z_j} \left(\frac{g(z_j)-\alpha}{1-\bar{\alpha} g(z_j)} \right) \right]^2 .$$

Since $\sum_{k=0}^{n} \gamma_k = 0$, we have

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \gamma_j \gamma_k \log |f(z_j)| = \left(\sum_{j=0}^{n} \gamma_j \log |f(z_j)| \right) \left(\sum_{k=0}^{n} \gamma_k \right) = 0 ,$$

and similarly for all other terms in the product on the left that involve only one index. The resulting simplification gives the desired inequality.

COROLLARY 5.3. *Suppose that g is analytic and univalent in $H = \{s : \operatorname{Re} s > 0\}$ and satisfies $g(H) \subset H$ and $\lim_{x \to \infty} \frac{g(x)}{x} = c > 0$. If $s_1, s_2, \ldots, s_n$ are in H and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are real then*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_j \gamma_k \log \left| \frac{(g(s_j)-g(s_k))(g(s_j)+\overline{g(s_k)})}{(s_j - s_k)(s_j + \bar{s}_k)} \right| \geqslant 2\gamma^2 \log c .$$

where $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n$.

Proof. Since the conclusion is not affected by adding an imaginary constant to g, we assume that $g(1) = \alpha > 0$. In order to use Theorem 5.1, let $s(z) = \frac{1+z}{1-z}$ and $f(z) = \frac{g(s(z))-\alpha}{g(s(z))+\alpha}$ for z in D. In addition, let $z_0, z_1, \ldots, z_n$ be defined by $s_j = s(z_j)$, where $s_0 > 0$ and let $\gamma_0 = -\gamma = -\sum_{j=1}^{n} \gamma_j$. The function f is univalent in D,

$f(D) \subset D$ and $f(0) = 0$, so that Theorem 5.1 applies. Making the substitutions we obtain

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \gamma_j \gamma_k \log \left| \frac{f'(0)(s_j-1)(s_k-1)(s_j+1)(\bar{s}_k+1)\alpha^2\big(g(s_j)-g(s_k)\big)\big(g(s_j)+\overline{g(s_k)}\big)}{(s_j-s_k)(s_j+\bar{s}_k)\big(g(s_j)-\alpha\big)\big(g(s_k)-\alpha\big)\big(\overline{g(s_k)}+\alpha\big)\big(g(s_j)+\alpha\big)} \right| \geqslant 0 .$$

Since $\sum_{k=0}^{n} \gamma_k = 0$, we simplify as in the previous corollary to get

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \gamma_j \gamma_k \log \left| \frac{\big(g(s_j)-g(s_k)\big)\big(g(s_j)+\overline{g(s_k)}\big)}{(s_j-s_k)(s_j+\bar{s}_k)} \right| \geqslant 0 .$$

Expanding this to isolate s_0, we have

$$\gamma^2 \log \left| g'(s_0) \frac{g(s_0)+\overline{g(s_0)}}{2s_0} \right| - 2\gamma \sum_{j=1}^{n} \gamma_j \log \left| \frac{\big(g(s_j)-g(s_0)\big)\big(g(s_j)+\overline{g(s_0)}\big)}{(s_j-s_0)(s_j+s_0)} \right|$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_j \gamma_k \log \left| \frac{\big(g(s_j)-g(s_k)\big)\big(g(s_j)+\overline{g(s_k)}\big)}{(s_j-s_k)(s_j+\bar{s}_k)} \right| \geqslant 0 .$$

Taking the limit as $s_0 \to \infty$ yields

$$\gamma^2 \log c^2 - 2\gamma \sum_{j=1}^{n} \gamma_j \log c^2 + \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_j \gamma_k \log \left| \frac{\big(g(s_j)-g(s_k)\big)\big(g(s_j)+\overline{g(s_k)}\big)}{(s_j-s_k)(s_j+\bar{s}_k)} \right| \geqslant 0$$

which $\left(\text{since } \gamma = \sum_{j=1}^{n} \gamma_j\right)$ is equivalent to our conclusion.

6. *Univalent functions: the case when $|a| = 1$*

In this section we prove inequalities for univalent functions whose Denjoy–Wolff point a lies on the unit circle. The first inequality deals with the case when $|\phi'(a)| < 1$ and is complete in the sense that necessary and sufficient conditions are given for equality.

THEOREM 6.1. *Let ϕ be analytic and univalent in D with $\phi(D) \subset D$ and suppose that the Denjoy–Wolff point a of ϕ satisfies $|a| = 1$ and $\phi'(a) < 1$. If $z_1, z_2, \ldots, z_n$ are distinct fixed points of ϕ on ∂D, different from a, then*

$$\sum_{j=1}^{n} \big(\log \phi'(z_j)\big)^{-1} \leqslant -\big(\log \phi'(a)\big)^{-1} .$$

Moreover, equality holds if and only if

$$\phi(z) = s^{-1}\Big(\sigma^{-1}\Big(\sigma\big(s(z)\big)+1\Big)\Big)$$

where

$$s(z) = (a+z)(a-z)^{-1} \qquad \text{and} \qquad \sigma(s) = \sum_{j=1}^{n} \big(\log \phi'(z_j)\big)^{-1} \log\big(s-s(z_j)\big) .$$

Proof. We shall use the Grunsky-type inequality, Corollary 5.3 of the last section. To do this we define g on the halfplane $H = \{s : \operatorname{Re} s > 0\}$ by

$$g(s) = \Big(a+\phi(z(s))\Big)\Big(a-\phi(z(s))\Big)^{-1}$$

where $s = (a+z)(a-z)^{-1}$. Then g is analytic and univalent in H and satisfies $g(H) \subset H$ and $\lim_{x\to\infty} \big(g(x)/x\big) = c = \phi'(a)^{-1}$. Moreover, g has fixed points $\zeta_1, \ldots, \zeta_n$ on the imaginary axis where $\zeta_j = s(z_j)$ and $g'(\zeta_j) = \phi'(z_j)$. Now let s_0 be an arbitrary point of H, let $s_j = x+\zeta_j$ for $x > 0$ and let $\gamma_0, \gamma_1, \ldots, \gamma_n$ be arbitrary real numbers with $\gamma = \gamma_1+\ldots+\gamma_n$. Then the inequality of Corollary 5.3 becomes

$$\gamma_0^2 \log\left|g'(s_0)\frac{\operatorname{Re} g(s_0)}{\operatorname{Re} s_0}\right| + 2\gamma_0 \sum_{j=1}^{n} \gamma_j \log\left|\left(\frac{g(s_j)-g(s_0)}{s_j-s_0}\right)\left(\frac{g(s_j)+\overline{g(s_0)}}{s_j+\bar{s}_0}\right)\right|$$

$$+ \sum_{j=1}^{n}\sum_{k=1}^{n} \gamma_j\gamma_k \log\left|\left(\frac{g(s_j)-g(s_k)}{s_j-s_k}\right)\left(\frac{g(s_j)+\overline{g(s_k)}}{s_j+\bar{s}_k}\right)\right| \geqslant 2(\gamma_0+\gamma)^2 \log c\,.$$

Taking the limit as x tends to zero we obtain

$$\gamma_0^2 \log\left|g'(s_0)\frac{\operatorname{Re} g(s_0)}{\operatorname{Re} s_0}\right| + 4\gamma_0 \sum_{j=1}^{n} \gamma_j \log\left|\frac{\zeta_j-g(s_0)}{\zeta_j-s_0}\right|$$

$$+2\sum_{j=1}^{n} \gamma_j^2 \log g'(\zeta_j) \geqslant 2(\gamma_0+\gamma)^2 \log c\,.$$

Taking $\gamma_j = \big(\log g'(\zeta_j)\big)^{-1}$ and collecting the terms with γ_0 we get

$$\frac{1}{2}\gamma_0^2 \log\left|\frac{g'(s_0)\operatorname{Re} g(s_0)}{c^2 \operatorname{Re} s_0}\right| + 2\gamma_0 \sum_{j=1}^{n} \gamma_j \log\left|\frac{\zeta_j-g(s_0)}{c(\zeta_j-s_0)}\right| + \gamma - \gamma^2 \log c \geqslant 0\,.$$

Since γ_0 is an arbitrary real number this means that

$$\gamma - \gamma^2 \log c \geqslant \frac{\left(\sum_{j=1}^{n} \gamma_j \log\left|\frac{\zeta_j-g(s_0)}{c(\zeta_j-s_0)}\right|\right)^2}{\frac{1}{2}\log\left|\frac{g'(s_0)\operatorname{Re} g(s_0)}{c^2 \operatorname{Re} s_0}\right|} \geqslant 0\,.$$

The inequality $\gamma-\gamma^2\log c \geqslant 0$ is just $\gamma \leqslant (\log c)^{-1}$, which is our conclusion.

Now if equality holds we see that $\operatorname{Re} \sum_{j=1}^{n} \gamma_j \log \dfrac{\zeta_j-g(s)}{c(\zeta_j-s)}$ is zero on H and since $\lim_{s\to\infty} \dfrac{\zeta_j-g(s)}{c(\zeta_j-s)} = 1$ we get

$$(*)\qquad \sum_{j=1}^{n} \gamma_j \log \frac{\zeta_j-g(s)}{c(\zeta_j-s)} \equiv 0 \quad \text{on } H\,.$$

If we define σ on H by $\sigma(s) = \sum_{j=1}^{n} \gamma_j \log(s-\zeta_j)$ then $\operatorname{Re}\sigma'(s) = \sum_{j=1}^{n} \gamma_j \frac{\operatorname{Re} s}{|s-\zeta_j|^2} > 0$ on H (a convex domain) and so σ is univalent. In fact, $\sigma(H)$ is the horizontal strip $\{w : |\operatorname{Im} w| < \gamma\pi/2\}$ with $n-1$ rays removed. The equation (*) says that g satisfies Abel's functional equation $\sigma(g(s)) = \sigma(s)+\gamma \log c = \sigma(s)+1$, which is what we were to prove.

Conversely, suppose that $\gamma_1, \gamma_2, \ldots, \gamma_n$ are positive numbers, that $\gamma = \sum_{j=1}^{n} \gamma_j$, and that $a, z_1, \ldots, z_n$ are distinct points of ∂D. If $\phi(z) = s^{-1}(\sigma^{-1}(\sigma(s(z))+1))$ where $s(z) = (a+z)(a-z)^{-1}$ and $\sigma(s) = \sum_{j=1}^{n} \gamma_j \log(s - s(z_j))$, then it is not difficult to check that ϕ is univalent in D with $\phi(D) \subset D$, the points $a, z_1, \ldots, z_n$ are fixed points of ϕ, and $\phi'(a) = e^{-1/\gamma}$ and $\phi'(z_j) = e^{1/\gamma_j}$.

REMARK. The equality condition above has a qualitative statement: Equality holds in the inequality of Theorem 6.1 if and only if $\phi(D)$ is the disk with $n-1$ analytic arcs removed and ϕ can be embedded in a continuous semigroup of iterates $\{\phi_t\}$ mapping D conformally into itself. Indeed, when equality holds, $\phi_t(z) = s^{-1}(\sigma^{-1}(\sigma(s(z))+t))$ is the required semigroup of iterates. The $n-1$ arcs comprising $D \setminus \phi(D)$ are the images under $s^{-1} \circ \sigma^{-1}$ of the line segments that make up $\sigma(H) \setminus (\sigma(H)+1)$. Conversely, if ϕ satisfies the semigroup conditions, then Theorems 3.4 and 5.2 of [3, pp. 81, 92] show that ϕ has the form above.

Before proving an inequality covering the case when $|a| = 1$, $|\phi'(a)| = 1$, we need to define some parameters that arise in the theorem. If ϕ is univalent on D and $\phi(D) \subset D$ the function $\log \frac{1-|\phi(0)|^2}{\phi'(0)z} \frac{\phi(z)-\phi(0)}{1-\overline{\phi(0)}\phi(z)} = \log(1+\ldots)$ is single valued on D. Let $l(z)$ be the determination of this function that has $l(0) = 0$. If the Denjoy–Wolff point of ϕ is 1 and $z_1, z_2, \ldots, z_n$ are the fixed points of ϕ on ∂D, let $b_j = \lim_{r \to 1^-} \operatorname{Im}(l(rz_j) - l(r))$. Thus b_j is a value of $\arg \frac{(1-\phi(0)\overline{z_j})(1-\overline{\phi(0)})}{(1-\overline{\phi(0)}\,z_j)(1-\phi(0))}$.

THEOREM 6.2. *Let ϕ be analytic and univalent in D with $\phi(D) \subset D$ and suppose that 1 is the Denjoy–Wolff point of ϕ and that $\phi'(1) = 1$. If $z_1, z_2, \ldots, z_n$ are distinct fixed points of ϕ on ∂D different from 1 and if $b_1, b_2, \ldots, b_n$ are as defined above, then*

$$\sum_{j=1}^{n} b_j^2 (\log \phi'(z_j))^{-1} \leqslant 2 \log \frac{1-|\phi(0)|^2}{|\phi'(0)|}.$$

Proof. Apply Corollary 5.2 with the points $rz_0 = r, rz_1, \ldots, rz_n$ for $0 < r < 1$ and take the limit as r tends to 1 to obtain

$$\sum_{j=1}^{n} \gamma_j^2 \log \phi'(z_j)^2 \geqslant \left(\log \frac{1-|\phi(0)|^2}{|\phi'(0)|}\right)^{-1} \left[\sum_{j=0}^{n} \gamma_j \lim_{r \to 1^-} l(rz_j)\right]^2.$$

But since $\gamma_0 = -\sum_{j=1}^{n} \gamma_j$ and since $\phi'(z_0) = \phi'(1) = 1$, this is equivalent to

$$2 \sum_{j=1}^{n} \gamma_j^2 \log \phi'(z_j) \geqslant \left(\log \frac{1-|\phi(0)|^2}{|\phi'(0)|}\right)^{-1} \left[\sum_{j=1}^{n} \gamma_j b_j\right]^2$$

where the $\gamma_1, \gamma_2, \ldots, \gamma_n$ are arbitrary real numbers. Choosing $\gamma_j = b_j(\log \phi'(z_j))^{-1}$ yields the given inequality.

It is quite unlikely that this is the best possible inequality: we have made a convenient but suboptimal choice for the γ_j. In addition, if ϕ is real valued on $(-1, 1)$ and ± 1 are fixed points with $\phi'(1) = 1$, the given inequality says nothing about $\phi'(-1)$ since $\lim_{r \to 1^-} \operatorname{Im}(l(-r)) = 0$. In this case, $\phi'(-1)$ undoubtedly depends on $\phi(0)$ and perhaps on other factors, but we cannot quantify the relationship.

7. *Univalent functions: the case when $a = 0$*

In this section we consider univalent functions $\phi(z) = bz + \ldots$ and obtain an inequality analogous to that of Theorem 6.1. We begin by finding an appropriate value of $\log b^{-1}$ for use in our inequality.

LEMMA 7.0. *Suppose that $\phi(z) = bz + \ldots$ is analytic and univalent in D with $\phi(D) \subset D$. If $z_1, z_2, \ldots, z_n$ are fixed points on ∂D then $\lim_{r \to 1^-} \log\left(\frac{\phi(rz_j)}{rz_j}\right)$ is the same for $j = 1, 2, \ldots, n$, where $\log\left(\frac{\phi(z)}{z}\right) = \log(b + \ldots)$ has the value* $\operatorname{Log} b$ *at $z = 0$.*

Proof. Since $\lim_{r \to 1^-} \phi(rz_j) = z_j$, we see that $\lim_{r \to 1^-} \left(\frac{\phi(rz_j)}{rz_j}\right) = 2\pi i k_j$. Now k_j is the number of times the curve $\phi(rz_j)$, $0 \leqslant r \leqslant 1$, winds around the origin in going from 0 to z_j. Since ϕ is univalent, these curves only intersect at 0, thus k_j is the same for each j.

THEOREM 7.1. *Let $\phi(z) = bz + \ldots$ be analytic and univalent in D with $\phi(D) \subset D$. If $z_1, z_2, \ldots, z_n$ are distinct fixed points of ϕ on ∂D, then*

$$\sum_{j=1}^{n} (\log \phi'(z_j))^{-1} \leqslant 2\operatorname{Re}(B^{-1})$$

where $B = \lim_{r \to 1^-} \log\left(\frac{\phi(rz_1)}{brz_1}\right)$. Moreover, equality holds if and only if $\phi(z) = \sigma^{-1}(b\sigma(z))$ where

$$\sigma(z) = z \prod_{j=1}^{n} (1 - \bar{z}_j z)^{-2\gamma_j/\gamma(1 - i\alpha^{-1}\beta)}$$

with $\gamma_j = (\log \phi'(z_j))^{-1}$ and $\sum_{j=1}^{n} \gamma_j = \gamma = 2\operatorname{Re} B^{-1}$, where $B = \alpha + i\beta$.

In the above, we choose the branch of $\log\left(\frac{\phi(z)}{bz}\right)$ that is zero at $z = 0$. Note that B is a particular value of $\log(\phi'(0)^{-1})$.

Proof. Let $z_0 = z$ be a point of D and let $\lambda = \gamma_0, \gamma_1, \dots, \gamma_n$ be real numbers with $\gamma = \sum_{j=1}^{n} \gamma_j$. For $0 < r < 1$ we have, from Theorem 5.1,

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \gamma_j \gamma_k \log \left| b \frac{rz_j rz_k(\phi(rz_j) - \phi(rz_k))(1 - \phi(rz_j)\overline{\phi(rz_k)})}{\phi(rz_j)\phi(rz_k)(rz_j - rz_k)(1 - r^2 z_j \bar{z}_k)} \right|$$
$$\geqslant (\log |b|^{-1})^{-1} \left[\sum_{j=0}^{n} \gamma_j \arg \frac{\phi(rz_j)}{brz_j} \right]^2 .$$

Now by the lemma, for $j = 1, \dots, n$, we have

$$B = \lim_{r \to 1-} \log \frac{\phi(rz_j)}{brz_j} = \log \frac{1}{|b|} + i \lim_{r \to 1-} \arg \frac{\phi(rz_j)}{brz_j} .$$

Writing $B = \alpha + i\beta$, and taking the limit as r tends to 1, we obtain

$$(*) \qquad \lambda^2 \log \left| \frac{bz^2 \phi'(z)(1 - |\phi(z)|^2)}{\phi(z)^2(1 - |z|^2)} \right| + 2\lambda \sum_{j=1}^{n} \gamma_j \log \left| \frac{bz(1 - \bar{z}_j \phi(z))^2}{\phi(z)(1 - \bar{z}_j z)^2} \right|$$
$$+ \sum_{j=1}^{n} \gamma_j^2 \log |b\phi'(z_j)^2| + 2 \sum_{1 \leqslant j < k \leqslant n} \gamma_j \gamma_k \log |b| \geqslant \alpha^{-1} \left[\lambda \arg \frac{\phi(z)}{bz} + \beta\gamma \right]^2 .$$

When $\lambda = 0$, we obtain $2 \sum_{j=1}^{n} \gamma_j^2 \log \phi'(z_j) - \gamma^2 \alpha \geqslant \alpha^{-1} \beta^2 \gamma^2$. Choosing $\gamma_j = (\log \phi'(z_j))^{-1}$, this becomes

$$2\gamma - \gamma^2 \alpha \geqslant \alpha^{-1} \beta^2 \gamma^2 \quad \text{or} \quad \gamma \leqslant 2(\alpha + \alpha^{-1} \beta^2)^{-1} = 2 \operatorname{Re} B^{-1} ,$$

which is the desired inequality.

If equality holds in this inequality, $(*)$ becomes

$$\lambda^2 \left[\log \left| \frac{bz^2 \phi'(z)(1 - |\phi(z)|^2)}{\phi(z)^2(1 - |z|^2)} \right| - \alpha^{-1} \left(\arg \left(\frac{\phi(z)}{bz} \right) \right)^2 \right]$$
$$+ 2\lambda \left[\sum_{j=1}^{n} \gamma_j \log \left| \frac{bz(1 - \bar{z}_j \phi(z))^2}{\phi(z)(1 - \bar{z}_j z)^2} \right| - \alpha^{-1} \beta\gamma \arg \frac{\phi(z)}{bz} \right] \geqslant 0 .$$

Since this holds for all λ, it follows that

$$\sum_{j=1}^{n} \gamma_j \log \left| \frac{bz(1 - \bar{z}_j \phi(z))^2}{\phi(z)(1 - \bar{z}_j z)^2} \right| - \alpha^{-1} \beta\gamma \arg \frac{\phi(z)}{bz} = 0 ,$$

for all z in D.

Since the analytic function of which this is the real part must be constant, we obtain

$$(1) \qquad 2 \sum_{j=1}^{n} \gamma_j \log \left(\frac{1 - \bar{z}_j \phi(z)}{1 - \bar{z}_j z} \right) - \gamma(1 - i\alpha^{-1}\beta) \log \frac{\phi(z)}{z} = \text{const.}$$

Now let

$$\sigma(z) = z \prod_{j=1}^{n} (1-\bar{z}_j z)^{-2\gamma_j/\gamma(1-i\alpha^{-1}\beta)}.$$

Using (1) and the definition of σ, we see that

$$\begin{aligned}\log \sigma(\phi(z)) &= \log \phi(z) - 2(\gamma(1-i\alpha^{-1}\beta))^{-1} \sum_{j=1}^{n} \gamma_j \log(1-\bar{z}_j \phi(z)) \\ &= \text{const} + \log z - 2(\gamma(1-i\alpha^{-1}\beta))^{-1} \sum_{j=1}^{n} \gamma_j \log(1-\bar{z}_j z).\end{aligned}$$

This means that $\sigma(\phi(z)) = b\sigma(z)$ (where the constant is obtained from the expansion at 0). Conversely, if $\gamma_1, \gamma_2, \ldots, \gamma_n$ are positive real numbers, $B = \alpha + i\beta$ is a complex number such that $\gamma = \sum_{j=1}^{n} \gamma_j = 2\operatorname{Re} B^{-1}$ and $z_1, z_2, \ldots, z_n$ are distinct points of ∂D, then we must show that the ϕ defined in the conclusion is a univalent map of D into itself with fixed points $0, z_1, z_2, \ldots, z_n$ and $\phi'(0) = e^{-B} = b$ and $\phi'(z_j) = e^{1/\gamma_j}$.

We examine the function σ. A computation gives

$$\begin{aligned}(1-i\alpha^{-1}\beta)z\frac{\sigma'(z)}{\sigma(z)} &= (1-i\alpha^{-1}\beta) + 2\gamma^{-1} \sum_{j=1}^{n} \gamma_j \bar{z}_j z(1-\bar{z}_j z)^{-1} \\ &= -i\alpha^{-1}\beta + \gamma^{-1} \sum_{j=1}^{n} \gamma_j \frac{1+\bar{z}_j z}{1-\bar{z}_j z},\end{aligned}$$

which means that

$$\operatorname{Re}\left[(1-i\alpha^{-1}\beta)z\frac{\sigma'(z)}{\sigma(z)}\right] = \gamma^{-1} \sum_{j=1}^{n} \gamma_j \frac{1-|z|^2}{|1-\bar{z}_j z|^2} > 0.$$

Theorem 6.6 of [7, p. 172] implies that σ is univalent in D and is spiral-like of type $\theta = \operatorname{Arg}(1-i\alpha^{-1}\beta)$, that is, that w in $\sigma(D)$ implies that $e^{-Bt}w$ is in $\sigma(D)$ for $t \geqslant 0$ [7, p. 171]. Thus, setting $t = 1$, we see that $b\sigma(z)$ is in $\sigma(D)$ for each z in D so that ϕ is a univalent map of D into itself with fixed points $0, z_1, z_2, \ldots, z_n$. A computation shows that $\phi'(0) = b$ and $\phi'(z_j) = e^{1/\gamma_j}$.

REMARK. The equality condition has a qualitative statement: Equality holds in the inequality of Theorem 7.1 if and only if $\phi(D)$ is the disk with n analytic arcs removed and ϕ can be embedded in a continuous semigroup of iterates $\{\phi_t\}$ mapping the disk conformally into itself with $(\phi_t)'(0) = e^{-Bt}$. Indeed, when equality holds $\phi_t(z) = \sigma^{-1}(e^{-Bt}\sigma(z))$ is the required semigroup of iterates (this is well defined since σ is spiral-like).

Since the imaginary part of $(1-i\alpha^{-1}\beta)\log\sigma(z)$ is constant on each of the n components of $\partial D \setminus \{z_j\}_{j=1}^{n}$, the image of σ is the plane with n spirals removed. This means that $\phi(D)$ is the disk with n analytic arcs removed, namely, the images under σ^{-1} of the parts of the n spirals $\sigma(D) \setminus e^{-B}\sigma(D)$.

In order to prove the sufficiency of the condition we suppose that ϕ maps the disk onto the disk with n analytic arcs removed and ϕ can be embedded in a continuous semigroup $\{\phi_t\}$ with $(\phi_t)'(0) = e^{Bt}$. We shall show that

$$\sum_{j=1}^{n} (\log \phi'(z_j))^{-1} = 2\operatorname{Re} B^{-1}$$

and that

$$\lim_{r\to 1^-} \log\left(\frac{\phi(rz_1)}{e^{-B}rz_1}\right) = B.$$

We shall use the model for ϕ developed in [3]. Theorems 3.2 and 3.3 of [3, pp. 78, 81] produce a conformal map $\sigma : D \to \mathbb{C}$ (σ is univalent because ϕ is) such that $\phi(z) = \sigma^{-1}(e^{-B}\sigma(z))$. Theorem 5.2 [3, p. 92] shows that $\phi_t(z) = \sigma^{-1}(e^{-Bt}\sigma(z))$ for each $t \geqslant 0$. The condition that $\phi(D)$ is the disk with n analytic arcs removed means $\sigma(D)$ is $\mathbb{C}\setminus \bigcup_{j=1}^{n} \Gamma_j$ where the curves Γ_j are spirals $\Gamma_j(t) = \{\beta_j e^{Bt} : t \geqslant 0\}$ (here β_j is the image under σ of the point where the j-th removed arc in D meets ∂D). Let $\Gamma = \{0\} \cup \{\beta_1 e^{Bt} : t \in \mathbb{R}\}$ so that Γ is a spiral from 0 to ∞. Without loss of generality, we suppose that 1 is in $\mathbb{C}\setminus\Gamma$ and we write $\operatorname{Log} w$ for the branch of the logarithm in $\mathbb{C}\setminus\Gamma$ such that $\operatorname{Log} 1 = 0$.

By the construction, e^{-Bt} is in $\mathbb{C}\setminus\Gamma$ for all $t > 0$ and $\operatorname{Log} e^{-Bt} = -Bt$. Thus, $\operatorname{Log}(\mathbb{C}\setminus\Gamma)$ is a strip parallel to the line $-Bt$ with vertical dimension 2π. Elementary trigonometry gives the width of this strip to be $2\pi \operatorname{Re} B|B|^{-1}$. The function $\psi(w) = -B^{-1}\operatorname{Log} w$ maps $\mathbb{C}\setminus\Gamma$ onto a strip of width $2\pi\operatorname{Re} B|B|^{-2}$. Now $\psi(\sigma(D)\setminus\Gamma)$ is a slit strip and we have $\phi(z) = \sigma^{-1}(\psi^{-1}(\psi(\sigma(z))+1))$ for z in $D\setminus\sigma^{-1}(\Gamma)$. If W_j is the distance between the rays that correspond to the fixed point z_j, we see, as in the computation in the proof of Theorem 3.4 of [3, p. 81], that $\phi'(z_j) = \exp(\pi/W_j)$. Thus $\sum_{j=1}^{n} (\log\phi'(z_j))^{-1} = \frac{1}{\pi}\sum_{j=1}^{n} W_j = 2\operatorname{Re} B|B|^{-2} = 2\operatorname{Re} B^{-1}$.

To complete the justification of the sufficiency of the condition, we must show that $\lim_{r\to 1^-} \log\left(\frac{\phi(rz_1)}{e^{-B}rz_1}\right) = B$. We note that $(z,t) \mapsto \phi_t(z)$ is a continuous function on $\bar{D}\times[0,1]$ which is a connected and simply connected set. Now define $L(z,t)$ on $\bar{D}\times[0,1]$ by $L(z,t) = \log\frac{e^{-Bt}\phi(z)}{e^{-B}\phi_t(z)}$ where we take $L(0,0) = 0$. We see that $L(z,0) = \log\frac{\phi(z)}{e^{-B}z}$, and so we want to compute $L(z_1, 0)$. We have $L(0,t) = 0$ for all t, so that $L(0,1) = 0$ and $L(z,1) = 0$ for all z in $\bar{D}$. In particular, $L(z_1, 1) = 0$. Since $\phi_t(z_1) = z_1$ for all t, this means that $L(z_1, t) = \log e^{B(1-t)} = B(1-t)$ so that $L(z_1, 0) = B$.

8. *Inequalities for* $\phi^{-1}(\{\lambda\})$

One interpretation of the inequalities on fixed points is that specifying values of ϕ places restrictions on the angular derivative. In this section we consider a variation in the way in which the values are specified: we examine what restrictions are placed on the derivative at points where ϕ has the value λ in ∂D. The first result treats general analytic functions.

THEOREM 8.1. *Let ϕ be analytic in D with $\phi(D) \subset D$ and suppose, for $j = 1, 2, \ldots, n$, that $\phi(z_j) = \lambda$ where $|z_j| = 1 = |\lambda|$. Then*

$$\sum_{j=1}^{n} \frac{1}{|\phi'(z_j)|} \leqslant \operatorname{Re} \frac{\lambda + \phi(0)}{\lambda - \phi(0)}$$

and equality holds if and only if ϕ is a Blaschke product of order n.

Proof. Let $g(z) = \lambda^{-1} z\phi(z)$, so that $0, z_1, \ldots, z_n$ are fixed points of g. Since $g'(0) = \lambda^{-1}\phi(0)$ and $g'(z_1) = 1 + \lambda^{-1} z_j \phi'(z_j) = 1 + |\phi'(z_j)|$, and g is a Blaschke product of order $n+1$ if and only if ϕ is a Blaschke product of order n, the result follows from Theorem 4.1(i).

The situation is somewhat easier for univalent functions. As is clear geometrically, we only need consider one point of $\phi^{-1}(\{\lambda\})$.

LEMMA 8.2. *Suppose that ϕ is analytic and univalent in D and that $\phi(D) \subset D$. If $\phi(z_j) = \lambda$ for $j = 1, 2, \ldots, n$ where $|z_j| = 1 = |\lambda|$, then $\phi'(z_j)$ is finite for at most one point z_j.*

Proof. Let $\Phi(z) = (\phi(z) - \phi(0))/(1 - \overline{\phi(0)}\phi(z))$, so that $\Phi(0) = 0$. Then $\Phi(z_1) = \ldots = \Phi(z_n)$ and $\Phi'(z_j)$ is finite if and only if $\phi'(z_j)$ is. Choose integers j and k, $1 \leqslant j < k \leqslant n$ and let $\gamma_j = \gamma_k = 1$ and $\gamma_m = 0$ for $m \neq j, k$. Theorem 5.1 shows that, for $0 < r < 1$,

$$\log \left| \Phi'(0)^2 \frac{r^2 z_j^2 r^2 z_k^2 \Phi'(rz_j)\Phi'(rz_k)(1 - |\Phi(rz_j)|^2)(1 - |\Phi(rz_k)|^2)}{\Phi(rz_j)^2 \Phi(rz_k)^2 (1 - r^2|z_j|^2)(1 - r^2|z_k|^2)} \right|$$

$$+ 2\log \left| \Phi'(0) \frac{r^2 z_j z_k (\Phi(rz_j) - \Phi(rz_k))(1 - \Phi(rz_j)\overline{\Phi(rz_k)})}{\Phi(rz_j)\Phi(rz_k)(rz_j - rz_k)(1 - r^2 z_j \overline{z_k})} \right|$$

$$\geqslant (\log|\Phi'(0)|^{-1})^{-1} \left(\arg \frac{\Phi(rz_j)\Phi(rz_k)}{\Phi'(0)^2 r^2 z_j z_k} \right)^2 \geqslant 0 .$$

Now since $\lim_{r \to 1^-} \Phi(rz_j) = \lim_{r \to 1^-} \Phi(rz_k)$, the second term tends to $-\infty$. The first term tends to $2\log|\Phi'(0)\Phi'(z_j)\Phi'(z_k)|$ which must be $+\infty$ in order to maintain the inequality. Since j and k are arbitrary, at most one of $\Phi'(z_1), \ldots, \Phi'(z_n)$ is finite.

The size of the angular derivative at this one λ-point is also restricted as we see in the following.

THEOREM 8.3. *Suppose that $\phi(z) = bz + \ldots$ is analytic and univalent in D and suppose that $\phi(z^*) = \lambda$ for $|z^*| = 1 = |\lambda|$. Then*

$$\frac{1}{\log|\phi'(z^*)|} \leqslant 2 \operatorname{Re} C^{-1}$$

where $C = \lim_{r \to 1^-} \log \frac{\phi(rz^*)}{brz^*}$, *which is a value of* $\log \frac{\lambda}{bz^*}$. *Moreover, equality holds if and only if*

$$\phi(z) = \frac{\lambda}{z^*} \sigma^{-1} \left(\frac{z^* b}{\lambda} \sigma(z) \right) \text{ where } \sigma(z) = z(1 - \bar{z}^* z)^{-2\,\mathrm{Re}\,C/\bar{C}}.$$

Proof. Let $\Phi(z) = z^* \lambda^{-1} \phi(z)$. Now $\Phi(z) = z^* \lambda^{-1} bz + \ldots$ is univalent, $\Phi(z^*) = z^*$, and $C = \lim_{r \to 1^-} \log \frac{\Phi(rz^*)}{(z^* \lambda^{-1} b) rz^*}$. Applying Theorem 7.1 gives the inequality, and the equality condition is obtained from the proof of 7.1.

In 8.3, ϕ is assumed to have a fixed point at zero as well as taking the value λ at z^*. If $\phi(z^*) = \lambda$ then we can use Theorem 4.1 to obtain

$$\phi'(z^*) \geqslant \left| \frac{a - \lambda}{a - z^*} \right|^2 \phi'(a)^{-1}$$

with equality if and only if ϕ is a Möbius transformation. (Univalence of ϕ is not necessary here.) It seems clear that many results combining assumptions about the fixed point set and $\phi^{-1}(\{\lambda\})$ are possible with these methods, but we do not pursue the subject further.

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