REARRANGING THE ALTERNATING HARMONIC SERIES

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In most elementary courses that include a discussion of infinite series, the instructor states and sometimes proves Riemann's theorem, "A conditionally convergent series can be rearranged to sum to any real number." However, after presenting this theorem, even in an advanced calculus or elementary analysis course, the instructor may get the uneasy feeling that the point of the theorem has been lost, that the issues involved are too subtle for students still afraid of *infinite* series. Perhaps some examples would be helpful—that is the point of this note. We calculate, explicitly, the sums of rearranged alternating harmonic series for a large class of rearrangements. More important, we give an example of a rearrangement that can be summed for freshman calculus students and a technique for adding such series that can be the basis of a set of exercises for more advanced students. The results presented here are old (1883) results of Pringsheim ([3], or see [1, pp. 74–77], [2, pp. 96–98], or [5, p. 25]), but they are not as well known as they should be.

By the alternating harmonic series, we mean the series

$$\sum_{k=1}^{\infty}(-1)^{k+1}k^{-1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots$$

whose sum is $\ln 2$. We say that a series is a *simple rearrangement* of an alternating series if it is a rearrangement of the series and the subsequence of positive terms and the subsequence of negative terms are in their original order. For example,

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\cdots \tag{1}$$

is a simple rearrangement of the alternating harmonic series whereas

$$1+\frac{1}{7}-\frac{1}{4}+\frac{1}{3}-\frac{1}{2}\cdots$$

is not. If $\Sigma_{k=1}^{\infty}a_k$ is a simple rearrangement of the alternating harmonic series, let p_n be the number of positive terms in $\{a_1,a_2,\ldots,a_n\}$ and let α denote the asymptotic density of the positive terms in the rearrangement. That is, $\alpha=\lim_{n\to\infty}p_n/n$, if the limit exists. Thus, $\alpha=\frac{1}{2}$ for the unrearranged alternating harmonic series and $\alpha=\frac{2}{3}$ for rearrangement (1) above.

THEOREM 1 [3]. *A simple rearrangement of the alternating harmonic series converges to an extended real number if and only if* α, *the asymptotic density of the positive terms in the rearrangement, exists. Moreover, the sum of a rearrangement with density* α *is* $\ln 2+\frac{1}{2}\ln(\alpha(1-\alpha)^{-1})$.

We shall give a short proof of this theorem below, but first we work out some special cases in a more naive way.

The simplest example of adding a rearrangement is attributed by Manning to Laurent [2, page 98]. Laurent's rearrangement is

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots \tag{2}$$

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where $\alpha = \frac{1}{3}$. One easily justifies inserting parentheses to get

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}\cdots=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots\right),$$

so the sum of series (2) is $\frac{1}{2}\ln 2$. The trick of inserting parentheses makes this an attractive example, but generalizing it to find sums of other rearrangements is more difficult.

Rearrangement (1) can be handled using power series in the same way that one uses the Taylor series for $\ln(1+x)$ to show $\sum_{k=1}^{\infty}(-1)^{k+1}k^{-1}=\ln 2$. Let

$$f(x)=x+\frac{x^3}{3}-\frac{x^4}{2}+\frac{x^5}{5}+\frac{x^7}{7}-\frac{x^8}{4}\cdots.$$

Elementary estimates show that series (1) converges and we conclude by Abel's theorem [4, Theorem 8.2, page 160] that its sum is $\lim_{x\to 1^-}f(x)$. Since

$$f(x)=\frac{1}{2}\left[\ln(1+x)-\ln(1-x)+\ln(1-x^4)\right]=\frac{1}{2}\ln\left[(1+x)^2(1+x^2)\right],$$

the sum of rearrangement (1) is $\frac{3}{2}\ln 2$. More generally, this technique works for rearrangements in which blocks of n positive terms alternate with blocks of m negative terms. For this case one uses

$$f(x)=\frac{1}{2}\left[\ln(1+x^m)-\ln(1-x^m)+\ln(1-x^{2n})\right]$$
$$=\frac{1}{2}\ln\left[(1+x^m)(1+x^n)(1+x+\cdots+x^{n-1})(1+x+\cdots+x^{m-1})^{-1}\right].$$

Computations of this sort make interesting exercises because their rigorous analysis requires several standard techniques from the theory of power series. (In fact, it is possible to prove Theorem 1 from this by noting that the sum of a rearrangement is an increasing function of the asymptotic density α.)

Proof of Theorem 1. Suppose $\sum_{k=1}^{\infty}a_k$ is a simple rearrangement of the alternating harmonic series. Let p_n be as above and $q_n=n-p_n$ so that

$$\sum_{k=1}^{n}a_k=\sum_{j=1}^{p_n}(2j-1)^{-1}-\sum_{j=1}^{q_n}(2j)^{-1}.$$

For each positive integer n, let $E_n=(\sum_{k=1}^{n}k^{-1})-\ln n$. The sequence $(E_n)_{n=1}^{\infty}$ is a decreasing sequence of positive numbers whose limit γ is called Euler's constant.

Now

$$\sum_{j=1}^{q_n}(2j)^{-1}=\frac{1}{2}\sum_{j=1}^{q_n}j^{-1}=\frac{1}{2}\ln q_n+\frac{1}{2}E_{q_n},$$

and

$$\sum_{j=1}^{p_n}(2j-1)^{-1}=\sum_{j=1}^{2p_n}j^{-1}-\sum_{j=1}^{p_n}(2j)^{-1}=\ln(2p_n)+E_{2p_n}-\tfrac{1}{2}\ln p_n-\frac{1}{2}E_{p_n}.$$

Thus

$$\lim_{n\to\infty}\sum_{k=1}^{n}a_k=\lim_{n\to\infty}\left[\ln 2p_n-\frac{1}{2}\ln p_n-\frac{1}{2}\ln q_n+E_{2p_n}-\frac{1}{2}E_{p_n}-\frac{1}{2}E_{q_n}\right]$$
$$=\ln 2+\lim_{n\to\infty}\frac{1}{2}\ln(p_nq_n^{-1})+\gamma-\frac{1}{2}\gamma-\frac{1}{2}\gamma,$$
$$=\ln 2+\frac{1}{2}\ln\left(\lim_{n\to\infty}p_nq_n^{-1}\right).$$

That is, the series converges iff the limit on the right, which is $\ln 2+\frac{1}{2}\ln(\alpha(1-\alpha)^{-1})$, exists. ∎

We have seen that the sums of rearrangements of the alternating harmonic series depend only on the asymptotic density α. This behavior is in some sense specific to series like the harmonic series, as Theorem 2 indicates. Readers are invited to construct proofs for themselves or to consult Pringsheim's paper [3].

THEOREM 2 [3]. *Suppose* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of real numbers such that* $|a_1| \geqslant |a_2| \geqslant |a_3| \geqslant \cdots$, $\lim_{n\to\infty} a_n = 0$, *and* $a_{2k-1} > 0 > a_{2k}$ *for* $k = 1, 2, 3, \ldots$.

(i) *If* $\lim_{n\to\infty} n|a_n| = \infty$, *and if* S *is a real number, there is a simple rearrangement of the series* $\sum_{k=1}^{\infty} a_k$ *with asymptotic density* $\frac{1}{2}$ *whose sum is* S.

(ii) *If* $\lim_{n\to\infty} na_n = 0$, *if* $\sum_{k=1}^{\infty} b_k$ *is a simple rearrangement of the series* $\sum_{k=1}^{\infty} a_k$ *for which the asymptotic density* α *exists, and if* $0 < \alpha < 1$, *then* $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_k$.

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A STRONG CONVERSE TO GAUSS'S MEAN-VALUE THEOREM

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The theorem of Gauss in the title affirms that

$$h(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a + re^{i\theta})\,d\theta \qquad (1)$$

holds for all a in a region Ω, all $r > 0$ such that the closure of the disc $D(a,r) = \{z \in \mathbb{C} : |z - a| < r\}$ lies in Ω, and all functions h that are harmonic throughout Ω. Most books on function theory or potential theory prove this elementary result as well as the following converse due to Koebe [4]: *If* h *is continuous in the region* Ω *and* (1) *holds for all* a *and* r *such that* $\bar{D}(a,r) \subset \Omega$, *then* h *is harmonic in* Ω. In fact, the somewhat stronger version in which the equality is required to hold only at each a for some sequence $r_n(a) \to 0$ is often proved. What does not seem to be well known is that, when h is continuous on $\bar{\Omega}$, one radius suffices. This strong converse of Gauss's theorem is due to Kellogg [3] and is not trivial. However, for Dirichlet regions this strong converse is as easy to prove as Koebe's theorem and should be presented in elementary texts. The theorem for Dirichlet regions is due to Volterra [7] (with a supplemental hypothesis) and to Vitali [6] (where the supplemental hypothesis is removed). Here is their proof in modern dress, presented in dimension two, though the reader will see that it is valid in any dimension.

LEMMA. *Let* U *be a bounded open subset of the complex plane* $\mathbb{C}$ *and let* $f: \bar{U} \to \mathbb{R}$ *be continuous and for each* $a \in U$ *have the following* restricted mean-value property:

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\,d\theta \text{ for some } r = r(a) > 0 \text{ such that } \bar{D}(a,r) \subset U. \qquad (2)$$

Then $\max f(\bar{U}) = \max f(\partial U)$.

Proof. (Cf. Cimmino [1]) Let $M = \max f(\bar{U})$. It suffices to see that the closed subset $f^{-1}(M)$ of