# Rearranging the Alternating Harmonic Series

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Based on a paper by Cowen, Ken Davidson, and Robert Kaufman:

*Amer. Math. Monthly* 87(1980) 817–819.

Rediscovery of work of Pringsheim:

*Math. Annalen* 22(1883) 455–503.

Commutative Law for addition: $a + b = b + a$

More generally,

$$\begin{aligned} \text{sum}\{a_1, a_2, \cdots, a_n\} &= a_1 + a_2 + \cdots + a_n \\ &= a_n + a_1 + a_2 + \cdots + a_{n-1} \\ &= a_2 + a_1 + a_n + a_{n-1} + \cdots + a_3 \\ &= \text{etc.} \end{aligned}$$

That is, the Commutative Law says rearranging the summands in a finite sum does not change the total.

Perhaps surprisingly, rearranging the summands in an infinite sum $CAN$ change the total!!

Fact:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\cdots \ = \ \ln 2$$

Consider:

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots$$

$$= \ \left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots$$

$$= \ \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}+\cdots$$

$$= \ \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\right)=\frac{1}{2}\ln 2 \qquad\qquad !!!!$$

Fact:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots = \ln 2$$

Consider:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2}\ln 2 \qquad !!!!$$

**Goal:** Think more carefully about rearrangements and find the sums of simple rearrangements of the AHS.

We think of an infinite series as

"adding up an infinite number of numbers"

– actually it is more subtle than that.

**Definition.**

We say $\sum_{n=1}^{\infty} a_n = S$ if $\lim_{N\to\infty} S_N = S$ where

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N$$

Rearranging terms of a series changes the partial sums.

Conceivably, this could change the limit of the partial sums.

In the earlier example, the partial sums of the rearranged series were approximately half the partial sums of the original series.

A series is called *conditionally convergent*

if the series converges but $\sum |a_n|$ diverges.

It can be shown that if $\sum |a_n|$ converges, then every rearrangement of the series $\sum a_n$ has the same sum.

**Riemann's Theorem.**

A conditionally convergent series (of real numbers) can be rearranged to sum to any real number.

**Definition.**

We say the series $\sum b_m$ is a *rearrangement* of the series $\sum a_n$ if each term of the series $\sum b_m$ occurs as a term of the series $\sum a_n$ (and the same number of times) and vice versa.

*Outline of proof of Riemann's Theorem.*

Since the given series converges, the terms of the series are small.

Since the given series is conditionally convergent, the series of positive terms diverges and the series of negative terms diverges.

Given $S$, we want to form a rearrangement of the series that sums to $S$.

To begin, select enough of the positive terms (choosing largest first) until the partial sum is larger than $S$.
(This is possible!)

For the next terms of the series, choose enough of the negative terms (choosing the most negative terms first) to make the partial sum less than $S$. (This is possible!)

Continue, choosing the positive and negatives terms in order of decreasing size so that the partial sums swing to more than $S$, then less, then more, etc.

Since the terms of the series are small, the oscillations get smaller and smaller so that the sequence of partial sums of the rearranged series converges to $S$. ■

**Definition.**

A *simple rearrangement* of a series is a rearrangement of the series in which the positive terms of the rearranged series occur in the same order as the original series and the negative terms occur in the same order.

Thus,

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots$$

is a simple rearrangement of the AHS

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

but

$$\frac{1}{3} - \frac{1}{2} - \frac{1}{6} + 1 + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{12} - \frac{1}{10} + \frac{1}{9} + \cdots$$

is not.

**Goal.**

Describe the sum of every simple rearrangement of the Alternating Harmonic Series.

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To do this, we'll use power series.

A *power series (centered at* $0$*)* is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

If the series converges for any non-zero $x$, there is an $R > 0$ so that the series converges in the open interval $-R < x < R$. In this interval, the series can be differentiated and integrated term by term and the resulting series also converge in this open interval.

**Abel's Theorem.**

If $\sum a_n$ converges, and if $f(x) = \sum a_n x^n$, then

$$\sum a_n = \lim_{x \to 1^-} f(x)$$

Abel's Theorem and the results on integration and differentiation of series allow us to find sums of series like the AHS.

To sum

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

let

$$f(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8 + \cdots$$

This power series converges in the open interval $-1 < x < 1$.

Let $F(x) = f'(x)$ so that

$$\begin{aligned} F(x) &= f'(x) = 1 - \frac{1}{2}2x + \frac{1}{3}3x^2 - \frac{1}{4}4x^3 + \frac{1}{5}5x^4 - \frac{1}{6}6x^5 + \cdots \\ &= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9 + \cdots \\ &= \frac{1}{1+x} \end{aligned}$$

Since $f'(x) = \dfrac{1}{1+x}$, we can see $f(x) = \ln(1+x)$.

Now Abel's Theorem says

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \lim_{x \to 1^-} \ln(1+x) = \ln 2$$

We used a trick to get from

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$$

to

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2} \ln 2$$

What about the rearrangement

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

It is not so clear what trick might work for this.

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It is not so clear what trick might work for this.

To use Abel's Theorem to sum the AHS, it was nice to have $\frac{x^k}{k}$ because differentiating gave an easy result.

This might suggest the series

$$x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{5}x^5 + \frac{1}{7}x^7 - \frac{1}{4}x^4 + \frac{1}{9}x^9 + \frac{1}{11}x^{11} - \frac{1}{6}x^6 + \cdots$$

But this series is not really a power series $\cdots$

The series

$$x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{5}x^5 + \frac{1}{7}x^7 - \frac{1}{4}x^4 + \frac{1}{9}x^9 + \frac{1}{11}x^{11} - \frac{1}{6}x^6 + \cdots$$

is not really a power series because the terms are in the wrong order!

Thinking about the given series as

$$1 \qquad + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \qquad + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} \qquad + \frac{1}{11} - \frac{1}{6} + \cdots$$

suggests the power series

$$x + 0x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 + 0x^6 + \frac{1}{7}x^7 - \frac{1}{4}x^8 + \frac{1}{9}x^9 + 0x^{10} + \cdots$$

Moreover, the power series

$$f(x) = x + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 + \frac{1}{7}x^7 - \frac{1}{4}x^8 + \frac{1}{9}x^9 + \cdots$$

converges on $-1 < x < 1$

Since the series converges absolutely on the interval $-1 < x < 1$, we can rearrange the series to get

$$\begin{aligned} f(x) &= x + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 + \frac{1}{7}x^7 - \frac{1}{4}x^8 + \frac{1}{9}x^9 + \cdots \\ &= \left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots\right) \\ &\quad - \left(\frac{1}{2}x^4 + \frac{1}{4}x^8 + \frac{1}{6}x^{12} + \cdots\right) \end{aligned}$$

Let

$$g(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots$$

and let

$$h(x) = \frac{1}{2}x^4 + \frac{1}{4}x^8 + \frac{1}{6}x^{12} + \frac{1}{8}x^{16} + \cdots$$

As before, we get

$$
\begin{aligned}
G(x) &= g'(x) = 1 + \frac{1}{3}3x^2 + \frac{1}{5}5x^4 + \frac{1}{7}7x^6 + \frac{1}{9}9x^8 + \cdots \\
&= 1 + x^2 + x^4 + x^6 + x^8 + \cdots = \frac{1}{1-x^2} \\
&= \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}
\end{aligned}
$$

so $g(x) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$

Similarly

$$
\begin{aligned}
H(x) &= h'(x) = \frac{1}{2}4x^3 + \frac{1}{4}8x^7 + \frac{1}{6}12x^{11} + \frac{1}{8}16x^{15} + \cdots \\
&= 2x^3 + 2x^7 + 2x^{11} + 2x^{15} \cdots \\
&= 2x^3 \left(1 + x^4 + x^8 + \cdots\right) = \frac{2x^3}{1 - x^4}
\end{aligned}
$$

so $h(x) = -\dfrac{1}{2}\ln(1 - x^4)$

To recapitulate, to get the sum of the rearrangement

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots$$

we define

$$f(x)=x+\frac{1}{3}x^3-\frac{1}{2}x^4+\frac{1}{5}x^5+\frac{1}{7}x^7-\frac{1}{4}x^8+\cdots=g(x)-h(x)$$

This means

$$\begin{aligned} f(x) &= \frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)+\frac{1}{2}\ln(1-x^4) \\ &= \frac{1}{2}\left[\ln(1+x)-\ln(1-x)+\ln(1-x^4)\right] \\ &= \frac{1}{2}\ln\left(\frac{(1+x)(1-x^4)}{1-x}\right)=\frac{1}{2}\ln\big((1+x)(1+x+x^2+x^3)\big) \end{aligned}$$

Using Abel's Theorem

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\lim_{x\to 1^-}f(x)=\frac{1}{2}\ln(8)=\frac{3}{2}\ln 2$$

This idea works more generally:

For simple rearrangements in which blocks of $n$ positive terms alternate with blocks of $m$ negative terms, use

$$\begin{aligned} f(x) &= \frac{1}{2}\left[\ln(1+x^m) - \ln(1-x^m) + \ln(1-x^{2n})\right] \\ &= \frac{1}{2}\ln\left(\frac{(1+x^m)(1+x^n)(1+x+x^2+\cdots+x^{n-1})}{1+x+x^2+\cdots+x^{m-1}}\right) \end{aligned}$$

So the series has the sum

$$\lim_{x\to 1^-} f(x) = \frac{1}{2}\ln\left(\frac{4n}{m}\right) = \ln 2 + \frac{1}{2}\ln\left(\frac{n}{m}\right)$$

This calculation suggests that the sum of a simple rearrangement of the AHS depends on the relative frequency of the positive terms of the series.

**Definition.**

If $\sum a_n$ is a simple rearrangement of the Alternating Harmonic Series, let $p_k$ be the number of positive terms in the first $k$ terms, $\{a_1, a_2, a_3, \cdots, a_k\}$.

The *asymptotic density,* $\alpha$, of the positive terms in the rearrangement is

$$\alpha = \lim_{k \to \infty} \frac{p_k}{k} \qquad \text{if the limit exists.}$$

In the rearrangement $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \cdots$

| $k$ | $\{a_1, \cdots, a_k\}$ | $p_k$ |
|---|---|---|
| 1 | $\{1\}$ | 1 |
| 2 | $\{1, -\frac{1}{2}\}$ | 1 |
| 3 | $\{1, -\frac{1}{2}, -\frac{1}{4}\}$ | 1 |
| 4 | $\{1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}\}$ | 2 |
| 5 | $\{1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{6}\}$ | 2 |
| | Etc. | |

so the asymptotic density is $\dfrac{1}{3}$.

**Theorem.**

*A simple rearrangement of the Alternating Harmonic Series converges to an extended real number if and only if* $\alpha$*, the asymptotic density of the positive terms in the rearrangement, exists.*

*Moreover, the sum of a rearrangement with asymptotic density* $\alpha$ *is*

$$\ln 2 + \frac{1}{2}\ln\left(\frac{\alpha}{1-\alpha}\right)$$

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For example, in the usual arrangement

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$p_1 = 1,\ p_2 = 1,\ p_3 = 2,\ p_4 = 2,\ p_5 = 3,\ p_6 = 3,\ \cdots$

so the asymptotic density is $\quad \alpha = \lim_{k\to\infty} \frac{p_k}{k} = \frac{1}{2}$

and the sum of the series is

$$\ln 2 + \frac{1}{2}\ln\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right) = \ln 2 + \ln 1 = \ln 2$$

**Theorem.**

*A simple rearrangement of the Alternating Harmonic Series converges to an extended real number if and only if* $\alpha$*, the asymptotic density of the positive terms in the rearrangement, exists.*

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$$\ln 2 + \frac{1}{2} \ln \left( \frac{\alpha}{1-\alpha} \right)$$

Similarly, in the rearrangement

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

$p_1 = 1,\ p_2 = 2,\ p_3 = 2,\ p_4 = 3,\ p_5 = 4,\ p_6 = 4,\ \cdots$

so the asymptotic density is $\quad \alpha = \lim_{k\to\infty} \frac{p_k}{k} = \frac{2}{3}$

and the sum of the series is

$$\ln 2 + \frac{1}{2} \ln \left( \frac{\frac{2}{3}}{1-\frac{2}{3}} \right) = \ln 2 + \frac{1}{2} \ln 2 = \frac{3}{2} \ln 2$$

**Outline of the proof.**

Suppose $\sum_{n=1}^{\infty} a_n$ is a simple rearrangement of the AHS.

Let $p_k$ be the number of positive terms in $\{a_1, a_2, a_3, \cdots, a_k\}$. Then the number of negative terms in the first $k$ terms is $q_k = k - p_k$.

Thus, because the given series is a simple rearrangement of the AHS,

$$\sum_{n=1}^{k} a_n = \sum_{j=1}^{p_k} \frac{1}{2j-1} - \sum_{j=1}^{q_k} \frac{1}{2j}$$

For each positive integer $m$, let

$$E_m = \sum_{n=1}^{m} \frac{1}{n} - \ln m$$

The sequence $E_1$, $E_2$, $E_3$, $\cdots$ is a decreasing sequence of positive numbers whose limit, $\gamma \approx .5772$, is called *Euler's constant*.

Since

$$\sum_{n=1}^{m} \frac{1}{n} \;=\; \ln m \;+\; E_m$$

we see that

$$\sum_{j=1}^{q_k} \frac{1}{2j} \;=\; \frac{1}{2}\sum_{j=1}^{q_k} \frac{1}{j} \;=\; \frac{1}{2}\ln q_k + \frac{1}{2}E_{q_k}$$

and

$$\sum_{j=1}^{p_k} \frac{1}{2j-1} \;=\; \sum_{\ell=1}^{2p_k} \frac{1}{\ell} - \sum_{\ell=1}^{p_k} \frac{1}{2\ell}$$

$$= \; \left(\ln(2p_k) + E_{2p_k}\right) - \left(\frac{1}{2}\ln p_k + \frac{1}{2}E_{p_k}\right)$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{k} a_n &= \sum_{j=1}^{p_k} \frac{1}{2j-1} - \sum_{j=1}^{q_k} \frac{1}{2j} \\
&= \ln(2p_k) + E_{2p_k} - \frac{1}{2}\ln p_k - \frac{1}{2}E_{p_k} - \frac{1}{2}\ln q_k - \frac{1}{2}E_{q_k} \\
&= \ln 2 + \ln p_k - \frac{1}{2}\ln p_k - \frac{1}{2}\ln q_k \\
&\quad + E_{2p_k} - \frac{1}{2}E_{p_k} - \frac{1}{2}E_{q_k}
\end{aligned}$$

We have

$$\begin{aligned}\sum_{n=1}^{k} a_n &= \ln 2 + \ln p_k - \frac{1}{2}\ln p_k - \frac{1}{2}\ln q_k \\ &\quad + E_{2p_k} - \frac{1}{2}E_{p_k} - \frac{1}{2}E_{q_k} \\ &= \ln 2 + \frac{1}{2}\ln p_k - \frac{1}{2}\ln q_k + E_{2p_k} - \frac{1}{2}E_{p_k} - \frac{1}{2}E_{q_k} \\ &= \ln 2 + \frac{1}{2}\ln\left(\frac{p_k}{q_k}\right) + E_{2p_k} - \frac{1}{2}E_{p_k} - \frac{1}{2}E_{q_k}\end{aligned}$$

This is the exact (!!) formula for the $k^{th}$ partial sum of the rearranged series.

To find the sum of the rearranged series, we need to take the limit of the partial sums.

Note that

$$\frac{p_k}{q_k} = \frac{p_k}{k - p_k} = \frac{\frac{p_k}{k}}{1 - \frac{p_k}{k}}$$

so if $\alpha$ exists

$$\lim_{k \to \infty} \frac{p_k}{q_k} = \frac{\alpha}{1 - \alpha}$$

and $p_k/q_k$ does not have a limit if $\alpha$ does not exist.

Note also that

$$\lim_{k\to\infty} E_{2p_k} = \lim_{k\to\infty} E_{p_k} = \lim_{k\to\infty} E_{q_k} = \gamma$$

Putting all of this together, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \lim_{k\to\infty} \sum_{n=1}^{k} a_n \\
&= \lim_{k\to\infty} \left( \ln 2 + \frac{1}{2} \ln\left(\frac{p_k}{q_k}\right) + E_{2p_k} - \frac{1}{2}E_{p_k} - \frac{1}{2}E_{q_k} \right) \\
&= \ln 2 + \frac{1}{2} \ln\left(\frac{\alpha}{1-\alpha}\right) + \gamma - \frac{1}{2}\gamma - \frac{1}{2}\gamma \\
&= \ln 2 + \frac{1}{2} \ln\left(\frac{\alpha}{1-\alpha}\right) \qquad \blacksquare
\end{aligned}$$

Riemann's Theorem on rearranging conditionally convergent series is non-constructive. In contrast, we have seen that simple rearrangements of the AHS converge if and only if the asymptotic density of the positive terms exists and we have a formula for the sum of the series when the asymptotic density exists.

In particular, the sum of a simple rearrangement of the AHS depends only on the asymptotic density of the positive terms.

This is a special property: the only series like this whose sums depend only on the asymptotic density of the positive terms are approximately multiples of the AHS.

More precisely, suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers so that $a_{2k-1} > 0 > a_{2k}$ for $k = 1, 2, \cdots$ and

$$|a_1| \geq |a_2| \geq |a_3| \geq |a_4| \geq \cdots$$

- If $\lim_{n\to\infty} n|a_n| = \infty$ (that is, $a_n$ is big compared to $\frac{1}{n}$) and $S$ is any real number, there is a simple rearrangement of the series with asymptotic density $\frac{1}{2}$ with sum $S$.
- If $\lim_{n\to\infty} n|a_n| = 0$ (that is, $a_n$ is small compared to $\frac{1}{n}$) and $\sum b_m$ is a simple rearrangement of $\sum a_n$ with asymptotic density $0 < \alpha < 1$, then

$$\sum b_m = \sum a_n$$

**THANK YOU!**

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