$\zeta(3)$

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1. INTRODUCTION

The Riemann zeta function at integers $k \geq 2$ is defined as follows:

$$\zeta(k) = \sum_{n \geq 1} \frac{1}{n^k}$$

We are interested in the arithmetic nature (rational / irrational, algebraic / transcendental) of the $\zeta(k)$'s. For even k, it turns out that $\zeta(k)$ is a rational multiple of π^k. We discuss this fact - originally due to Euler - in Section 4. Since π is transcendental, it follows that $\zeta(k)$ is transcendental. For odd k, much less is known. Apéry [2] broke the ice in 1978 with a miraculous proof of the following:

Theorem (Apéry)**.** *$\zeta(3)$ is irrational.*

Whether $\zeta(3)$ is transcendent, or whether $\zeta(3)$ is a rational multiple of π^3, is unknown. Also, the irrationality of any specific value of the zeta function at odd integers greater than 3 is unknown. Two results in this direction are, however, worth mentioning. Rivoal [10] has shown the following:

Theorem (Rivoal)**.** *Infinitely many ζ-values at odd integers are irrational.*

In fact, we have ([6, Thm.1]): if $n \geq 3$ is an odd integer, then the dimension of the $\mathbb{Q}$-vector space generated by 1, $\zeta(3)$, $\zeta(5)$, $\ldots$, $\zeta(n)$ is at least $\frac{1}{3}\ln n$. It follows that there is an infinite subset of $\{1, \zeta(3), \zeta(5), \zeta(7), \ldots\}$ which is linearly independent over $\mathbb{Q}$. On one hand, at most one element of the infinite subset is rational, hence the previous theorem. On the other hand, this linear independence over $\mathbb{Q}$ fits the hypothesis that $\zeta(k)$ is a rational multiple of π^k for odd k, as well.

Another irrationality result for values of the zeta function at odd integers is due to Zudilin [12]:

Theorem (Zudilin)**.** *One of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.*

Apéry's theorem is the main result we are after in this paper. We follow an elegant proof due to Beukers [4]. Just like Apéry's original proof (see [9] for a lively exposition), Beukers' proof deals with both $\zeta(2)$ and $\zeta(3)$. The proof that $\zeta(2)$ is irrational (Section 2) is interesting here not for the result in itself, but rather for warming-up the ground for the slightly more involved proof that $\zeta(3)$ is irrational (Section 3).

The basic idea for showing that a given real number ξ is irrational is to construct a sequence of non-zero integral combinations $\{a_n + b_n\xi\}_{n\geq 1}$ $(a_n, b_n \in \mathbb{Z})$ which converges to 0. Indeed, if ξ were rational then $a_n + b_n\xi$ would be bounded away from zero independently of n.

Before proceeding to the proofs, let us prepare our tools. First, the Legendre polynomials

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}\left(x^n(1-x)^n\right) \qquad (n \geq 0)$$

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which are immediately seen to have integer coefficients. These polynomials are particularly friendly to integration by parts. For $i \leq n-1$ we have

$$\frac{d^i}{dx^i}\left(x^n(1-x)^n\right)(0) = \frac{d^i}{dx^i}\left(x^n(1-x)^n\right)(1) = 0$$

so for an integrable function g we can write:

$$\begin{aligned}\int_0^1 P_n(x)g(x)\,dx &= \int_0^1 \frac{1}{n!}\frac{d^n}{dx^n}\left(x^n(1-x)^n\right)\,g(x)\,dx\\ &= \left[\frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}\left(x^n(1-x)^n\right)\,g(x)\right]_0^1 - \int_0^1 \frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}\left(x^n(1-x)^n\right)\,g'(x)\,dx\\ &= -\int_0^1 \frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}\left(x^n(1-x)^n\right)\,g'(x)\,dx\end{aligned}$$

Repeating this process n times and taking the absolute value we obtain:

$$\left|\int_0^1 P_n(x)g(x)\,dx\right| = \left|\int_0^1 \frac{1}{n!}x^n(1-x)^n g^{(n)}(x)\,dx\right| \tag{1}$$

Second, consider the least common multiple of $1, 2, \ldots, n$:

$$d_n = \mathrm{lcm}(1, 2, \ldots, n)$$

We will need the following bound:

$$d_n < 3^n \quad (n \gg 1) \tag{2}$$

Indeed, notice that $d_n = \prod p^{\alpha_p}$, where the product is taken over all primes $p \leq n$ and each α_p is the greatest integer with $p^{\alpha_p} \leq n$. So $d_n \leq n^{\pi(n)}$, where $\pi(n)$ denotes the number of primes no greater than n. The prime number theorem says that

$$1 = \lim_{n\longrightarrow\infty} \frac{\pi(n)\ln n}{n} = \lim_{n\longrightarrow\infty} \frac{\ln n^{\pi(n)}}{n}$$

so for n sufficiently large we have $\ln n^{\pi(n)} < n\ln 3$, i.e., $d_n \leq n^{\pi(n)} < 3^n$ as desired.

Whenever it occurs, the interchange of integral with infinite sum is justified by the monotone convergence theorem, the interchange of integral with derivative is allowed since the functions involved are "nice", whereas the constant use of improper integrals instead of limits of proper ones is somehow excused by the "uncertainty principle of writing proofs": a gain in rigor is a loss in clarity.

2. MORNING WARM-UP: IRRATIONALITY OF $\zeta(2)$

Lemma 2.1. *For all* $0 \leq x, y \leq 1$ *we have:*

$$\frac{x(1-x)y(1-y)}{1-xy} \leq \left(\frac{\sqrt{5}-1}{2}\right)^5$$

Proof. Let $f(x,y)$ be the function given in the lemma. Notice first that f vanishes on the boundary of $[0,1]\times[0,1]$. The function f is not defined for $(1,1)$, but we have $f(x,y) \longrightarrow 0$ as $x, y \nearrow 1$.

To find the maximum of f in the unit square, we solve the following system in $(0,1)\times(0,1)$:

$$\frac{\partial}{\partial x}f(x,y) = 0 = \frac{\partial}{\partial y}f(x,y)$$

which immediately takes the form:

$$1 - 2x + yx^2 = 0 = 1 - 2y + xy^2$$

Express y from the first relation, substitute in the second relation and obtain $x^3 - 2x + 1 = 0$, whose roots are 1, $\frac{-1\pm\sqrt{5}}{2}$. Hence $x = \frac{\sqrt{5}-1}{2}$ and, by symmetry, $y = \frac{\sqrt{5}-1}{2}$; these are the coordinates of the point where f achieves its maximum value $\left(\frac{\sqrt{5}-1}{2}\right)^5$. $\square$

In what follows, we use $\iint_\square$ to mean the double integral over the unit square $[0,1] \times [0,1]$.

Proposition 2.2. *Let $r, s \in \mathbb{N}$. Then*

$$\iint_\square \frac{x^r y^s}{1-xy}\, dxdy \in \begin{cases} \zeta(2) + \frac{1}{d_r^2}\mathbb{Z} & \text{if } r = s \\ \frac{1}{d_r^2}\mathbb{Z} & \text{if } r > s \end{cases}$$

Proof. For any real $a \geq 0$ we have

$$\begin{aligned} \iint_\square \frac{x^{r+a} y^{s+a}}{1-xy}\, dxdy &= \iint_\square x^{r+a} y^{s+a} \sum_{n\geq 0} (xy)^n \, dxdy = \sum_{n\geq 0} \iint_\square x^{n+r+a} y^{n+s+a}\, dxdy \\ &= \sum_{n\geq 0} \frac{1}{n+r+a+1} \cdot \frac{1}{n+s+a+1} \end{aligned} \tag{3}$$

For the present proof, we only use (3) for $a = 0$, but the full force of (3) will be needed later, when dealing with $\zeta(3)$. If $r = s$, then for $a = 0$ we get:

$$\iint_\square \frac{x^r y^r}{1-xy}\, dxdy = \sum_{n\geq 0} \frac{1}{(n+r+1)^2} = \zeta(2) - \frac{1}{1^2} - \cdots - \frac{1}{r^2} \in \zeta(2) + \frac{1}{d_r^2}\mathbb{Z}$$

In particular, for $r = 0$ we have the integral representation:

$$\zeta(2) = \iint_\square \frac{dxdy}{1-xy} \tag{4}$$

If $r > s$, then we can express the sum in (3) as

$$\begin{aligned} \sum_{n\geq 0} \frac{1}{n+r+a+1} \cdot \frac{1}{n+s+a+1} &= \frac{1}{r-s} \sum_{n\geq 0} \left(\frac{1}{n+s+a+1} - \frac{1}{n+r+a+1} \right) \\ &= \frac{1}{r-s} \left(\frac{1}{s+a+1} + \cdots + \frac{1}{r+a} \right) \end{aligned} \tag{5}$$

Setting again $a = 0$, we get that the last sum can be expressed as a ratio whose denominator is d_r^2. $\square$

Theorem 2.3. *$\zeta(2)$ is irrational.*

Proof. Consider the following integral

$$\iint_\square \frac{P_n(x)(1-y)^n}{1-xy}\, dxdy$$

On one hand, this integral is of the form $(a_n + b_n\zeta(2))/d_n^2$ with $a_n, b_n \in \mathbb{Z}$, since P_n is a polynomial with integer coefficients. On the other hand, using (1) we have

$$\begin{aligned}\left|\iint_\square \frac{P_n(x)(1-y)^n}{1-xy}\,dxdy\right| &= \left|\int_0^1 P_n(x)\left(\int_0^1 \frac{(1-y)^n}{1-xy}dy\right)\,dx\right| \\ &= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\frac{d^n}{dx^n}\left(\int_0^1 \frac{(1-y)^n}{1-xy}\,dy\right)\,dx\right| \\ &= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\left(\int_0^1 \frac{d^n}{dx^n}\left(\frac{(1-y)^n}{1-xy}\right)\,dy\right)\,dx\right| \\ &= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\left(\int_0^1 \frac{n!y^n(1-y)^n}{(1-xy)^{n+1}}\,dy\right)\,dx\right| \\ &= \iint_\square \frac{x^n(1-x)^n y^n(1-y)^n}{(1-xy)^{n+1}}\,dxdy\end{aligned}$$

Hence the integral we considered does not vanish, and by Lemma 2.1 we have:

$$0 < \left|\frac{a_n + b_n\zeta(2)}{d_n^2}\right| \leq \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \iint_\square \frac{dxdy}{1-xy} = \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \zeta(2)$$

Using the bound (2) on d_n, for n sufficiently we have

$$0 < |a_n + b_n\zeta(2)| < 9^n \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \zeta(2) < 0.9^n$$

which implies that $\zeta(2)$ is irrational. □

We can actually compute $\zeta(2)$ by using a double-integral representation. It is not really (4) that we need, but a slightly modified formula:

$$\iint_\square \frac{dxdy}{1-x^2y^2} = \iint_\square \sum_{n\geq 0}(xy)^{2n}\,dxdy = \sum_{n\geq 0}\frac{1}{(2n+1)^2} = \frac{3}{4}\zeta(2)$$

To compute the above double integral, we will use a clever substitution:

$$A = \left\{u, v : u \geq 0,\ v \geq 0,\ u+v \leq \frac{\pi}{2}\right\} \longrightarrow [0,1]\times[0,1], \quad (u,v) \longrightarrow \left(\frac{\sin u}{\cos v}, \frac{\sin v}{\cos u}\right)$$

Some trigonometric gymnastics show that the above transformation is well-defined and it is bijective, its inverse being given by

$$(x,y) \longrightarrow \left(\arctan x\sqrt{\frac{1-y^2}{1-x^2}},\ \arctan y\sqrt{\frac{1-x^2}{1-y^2}}\right).$$

The Jacobian of the transformation is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos u/\cos v & \sin u \sin v/\cos^2 v \\ \sin u\sin v/\cos^2 u & \cos v/\cos u \end{vmatrix} = 1 - \frac{\sin^2 u \sin^2 v}{\cos^2 u \cos^2 v} = 1 - x^2y^2$$

We thus get

$$\frac{3}{4}\zeta(2) = \iint_\square \frac{dxdy}{1-x^2y^2} = \iint_A dudv = \text{Area}\,(A) = \frac{\pi^2}{8}$$

hence $\zeta(2) = \pi^2/6$.

A higher-dimensional version of this computation allows one to prove that $\zeta(2k)$ is a rational multiple of π^{2k}; see [5], as well as the exposition in [7, §1,2].

3. Noon climax: Irrationality of $\zeta(3)$

Lemma 3.1. *For all $0 \le x, y, w \le 1$ we have:*

$$\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} \le (\sqrt{2}-1)^4$$

Proof. Let $f(x,y,w)$ be the function given in the lemma. Notice that f vanishes on the boundary of the unit cube $[0,1]\times[0,1]\times[0,1]$. On the edges $x=0, w=1$ and $y=0, w=1$ the function f is not defined, but we have $f(x,y,w) \longrightarrow 0$ as $x \searrow 0, w \nearrow 1$ or as $y \searrow 0, w \nearrow 1$.

To find the maximum of f over the unit cube, we solve the following system in $(0,1)\times(0,1)\times(0,1)$:

$$\frac{\partial}{\partial x} f(x,y,w) = \frac{\partial}{\partial y} f(x,y,w) = \frac{\partial}{\partial w} f(x,y,w) = 0$$

which immediately takes the form:

$$1-2w+(1-xy)w^2 = (1-2x)-(1-2x+x^2y)w = (1-2y)-(1-2y+xy^2)w = 0$$

Equating the expressions for w from the last two relations gives $x=y$. Use this in the first relation and get $w = \frac{1}{1+x}$. The last two relations, now rendered equivalent, give $w = \frac{1-2x}{1-2x+x^3}$. Equating these two expressions quickly leads to $x^2+2x-1=0$ whose roots are $-1\pm\sqrt{2}$. Hence $x=y=\sqrt{2}-1$ and $w = 1/\sqrt{2}$; these are the coordinates of the point where f achieves its maximum value $(\sqrt{2}-1)^4$. □

Proposition 3.2. *Let $r,s \in \mathbb{N}$. Then*

$$\iint_\square -\frac{x^r y^s \ln xy}{1-xy}\,dxdy \in \begin{cases} 2\zeta(3)+\frac{1}{d_r^3}\mathbb{Z} & \text{if } r=s \\ \frac{1}{d_r^3}\mathbb{Z} & \text{if } r>s \end{cases}$$

Proof. If $r=s$, it follows by (3) that for all $a \ge 0$ we have:

$$\iint_\square \frac{x^{r+a}y^{r+a}}{1-xy}\,dxdy = \sum_{n\ge 0}\frac{1}{(n+r+a+1)^2}$$

Differentiate with respect to a to obtain:

$$\iint_\square \frac{x^{r+a}y^{r+a}\ln xy}{1-xy}\,dxdy = -2\sum_{n\ge 0}\frac{1}{(n+r+a+1)^3}$$

At $a=0$ this says

$$\iint_\square \frac{x^r y^r \ln xy}{1-xy}dxdy = -2\sum_{n\ge 0}\frac{1}{(n+r+1)^3} = -2\left(\zeta(3)-\frac{1}{1^3}-\cdots-\frac{1}{r^3}\right) \in -2\zeta(3)-\frac{1}{d_r^3}\mathbb{Z}.$$

In particular, for $r=0$ we have the integral representation:

$$\zeta(3) = -\frac{1}{2}\iint_\square \frac{\ln xy}{1-xy}\,dxdy$$

For $r>s$, recall that relation (5) says the following:

$$\iint_\square \frac{x^{r+a}y^{s+a}}{1-xy}\,dxdy = \frac{1}{r-s}\left(\frac{1}{s+a+1}+\cdots+\frac{1}{r+a}\right)$$

Differentiate with respect to a and obtain:

$$\iint_\square \frac{x^{r+a}y^{s+a}\ln xy}{1-xy}\,dxdy = \frac{-1}{r-s}\left(\frac{1}{(s+a+1)^2}+\cdots+\frac{1}{(r+a)^2}\right)$$

For $a=0$ we get

$$\iint_{\square} \frac{x^r y^s \ln xy}{1-xy}\,dxdy = \frac{-1}{r-s}\left(\frac{1}{(s+1)^2}+\cdots+\frac{1}{r^2}\right) \in \frac{1}{d_r^3}\mathbb{Z}$$

which ends the proof. □

Theorem 3.3. $\zeta(3)$ *is irrational.*

Proof. Consider the following integral:

$$\iint_{\square} -\frac{P_n(x)P_n(y)\ln xy}{1-xy}\,dxdy$$

On one hand, this integral equals $(a_n+b_n\zeta(3))/d_n^3$ for some $a_n, b_n \in \mathbb{Z}$. On the other hand, since

$$-\frac{\ln xy}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z}\,dz$$

we have

$$\begin{aligned}
\left|\iint_{\square} -\frac{P_n(x)P_n(y)\ln xy}{1-xy}\,dxdy\right| &= \left|\int_0^1 P_n(x)\left(\iint_{\square}\frac{P_n(y)}{1-(1-xy)z}\,dydz\right)dx\right| \\
&= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\frac{d^n}{dx^n}\left(\iint_{\square}\frac{P_n(y)}{1-(1-xy)z}\,dydz\right)dx\right| \\
&= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\left(\iint_{\square}\frac{d^n}{dx^n}\left(\frac{P_n(y)}{1-(1-xy)z}\right)dydz\right)dx\right| \\
&= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\left(\iint_{\square}\frac{(-1)^n n! P_n(y) y^n z^n}{(1-(1-xy)z)^{n+1}}\,dydz\right)dx\right| \\
&= \left|\int_0^1 P_n(y)\left(\iint_{\square}\frac{x^n(1-x)^n y^n z^n}{(1-(1-xy)z)^{n+1}}\,dxdz\right)dy\right|
\end{aligned}$$

Make the change of variables $w=\frac{1-z}{1-(1-xy)z}$ (an involution) so $dw=\frac{-xy}{(1-(1-xy)z)^2}\,dz$. We get:

$$\begin{aligned}
\left|\iint_{\square} -\frac{P_n(x)P_n(y)\ln xy}{1-xy}\,dxdy\right| &= \left|\int_0^1 P_n(y)\left(\iint_{\square}\frac{(1-x)^n(1-w)^n}{1-(1-xy)w}\,dxdw\right)dy\right| \\
&= \left|\int_0^1 \frac{y^n(1-y)^n}{n!}\frac{d^n}{dy^n}\left(\iint_{\square}\frac{(1-x)^n(1-w)^n}{1-(1-xy)w}\,dxdw\right)dy\right| \\
&= \left|\int_0^1 \frac{y^n(1-y)^n}{n!}\left(\iint_{\square}\frac{d^n}{dy^n}\left(\frac{(1-x)^n(1-w)^n}{1-(1-xy)w}\right)dxdw\right)dy\right| \\
&= \left|\int_0^1 \frac{y^n(1-y)^n}{n!}\left(\iint_{\square}\frac{(-1)^n n!(1-x)^n(1-w)^n x^n w^n}{(1-(1-xy)w)^{n+1}}\,dxdw\right)dy\right| \\
&= \iiint_0^1 \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{(1-(1-xy)w)^{n+1}}\,dxdydw
\end{aligned}$$

Thus the integral we considered does not vanish, and by Lemma 3.1 we obtain:

$$0 < \left|\frac{a_n+b_n\zeta(3)}{d_n^3}\right| \le (\sqrt{2}-1)^{4n}\iiint_0^1 \frac{dxdydw}{1-(1-xy)w} = (\sqrt{2}-1)^{4n}\iint_{\square} -\frac{\ln xy}{1-xy}\,dxdy$$

That is

$$0 < |a_n+b_n\zeta(3)| \le d_n^3(\sqrt{2}-1)^{4n}2\zeta(3)$$

By our bound (2) on d_n, for sufficiently large n we have

$$0 < |a_n+b_n\zeta(3)| < 27^n(\sqrt{2}-1)^{4n}2\zeta(3) < 0.8^n$$

which implies that $\zeta(3)$ is irrational. □

4. Evening contemplation: The zeta function at even integers

In this section, we outline several proofs for the following formula due to Euler:

$$\zeta(2k) = \frac{(-1)^{k+1}B_{2k}}{2(2k)!}(2\pi)^{2k} \quad (k \geq 1) \tag{6}$$

The Bernoulli numbers $(B_n)_{n\geq 0}$ which appear in formula (6) are defined by the Taylor expansion

$$\frac{x}{e^x-1} = \sum_{n\geq 0} B_n \frac{x^n}{n!}.$$

Focusing on the coefficient of x^n in

$$1 = \left(\frac{x}{e^x-1}\right)\left(\frac{e^x-1}{x}\right) = \left(\sum_{n\geq 0} B_n \frac{x^n}{n!}\right)\left(\sum_{n\geq 0} \frac{x^n}{(n+1)!}\right)$$

we reach the following recursion formula:

$$\binom{n+1}{1} B_n + \binom{n+1}{2} B_{n-1} + \cdots + \binom{n+1}{n} B_1 + B_0 = 0$$

In particular, the Bernoulli numbers are rational numbers (hence, once we establish (6), it will follow that $\zeta(2k)$ is a rational multiple of π^{2k}). Explicitly, the Bernoulli numbers start off as 1, $-\frac{1}{2}$, $\frac{1}{6}$, 0, $-\frac{1}{30}$, 0, $\frac{1}{42}$, 0, $-\frac{1}{30}$, 0, $\frac{5}{66}$,.... This indicates that $B_n = 0$ for odd $n > 1$, and this is indeed the case: the function

$$\frac{x}{e^x-1} + \frac{x}{2} = \frac{x}{2} \cdot \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}}$$

is even, so $B_1 = -1/2$ and $B_n = 0$ for odd $n \geq 3$. [1]

One way of obtaining (6) is through the Riemann functional equation. The zeta function is first defined on the open half-plane $\text{Re}(s) > 1$ (where the series converges absolutely) by

$$\zeta(s) = \sum_{n\geq 1} n^{-s}.$$

Using contour integration, one extends the zeta function to a function which is analytic throughout the complex plane, except for a simple pole at $s = 1$ with residue 1 (i.e., $\lim_{s\to 1}(s-1)\zeta(s) = 1$). Thus extended, the zeta function satisfies the Riemann functional equation

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

relating the values of ζ on arguments symmetric about the critical line $\text{Re}(s) = 1/2$. Here, the gamma function is defined for $\text{Re}(s) > 0$ as

$$\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du$$

and then extended to a function that is analytic in the whole complex plane, except simple poles at $0, -1, -2, -3, \ldots$. Note that $\Gamma(n+1) = n!$.

The zeta function takes a simple form on negative integers:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

[1] For even n, an interesting arithmetic information on the B_n's is given by the Clausen - von Staudt theorem: the denominator of B_n is $\prod_{\substack{p-1\mid n\\ p\text{ prime}}} p$.

We see, in particular, the trivial zeros of the zeta function: $\zeta(-n) = 0$ for even $n \geq 2$. On the other hand, set $s = 2k$ with k a positive integer in the functional equation for ζ to get

$$-\frac{B_{2k}}{2k} = \zeta(1-2k) = 2(2\pi)^{-2k}\Gamma(2k)\cos(\pi k)\zeta(2k) = 2(2\pi)^{-2k}(2k-1)!(-1)^k\zeta(2k)$$

that is, formula (6). Notice that no similar information about $\zeta(2k+1)$ can be obtained by plugging in $s = 2k+1$, since both sides of the functional equation will vanish.

Another way of deriving (6) starts from the following convolution formula for the $\zeta(2k)$'s:

$$(k + \frac{1}{2})\zeta(2k) = \sum_{i=1}^{k-1} \zeta(2i)\zeta(2k-2i) \qquad (k \geq 2) \tag{7}$$

See Williams [11] for a one-page proof using elementary series manipulation. Let us note that (7) gives that $\zeta(2k)$ is a rational multiple of $\zeta(2)^k$. As $\zeta(2) = \pi^2/6$, we can already conclude that $\zeta(2k)$ is a rational multiple of π^{2k}.

To obtain (6) from (7), we rely on the following

Lemma 4.1. *Let* $C_{2k} = \frac{1}{(2k)!}B_{2k}$. *Then* $-(2k+1)C_{2k} = \sum_{i=1}^{k-1} C_{2i}C_{2k-2i}$ *for* $k \geq 2$.

Proof. We have the following series expansion:

$$\frac{x}{e^x - 1} = -\frac{x}{2} + \sum_{k \geq 0} C_{2k}\, x^{2k} \tag{8}$$

We will compute in two ways the coefficient of x^{2k} for $k \geq 2$ in the expansion of $x^2/(e^x-1)^2$. On one hand, squaring (8) we obtain that the respective coefficient is $C_0C_{2k} + C_2C_{2k-2} + \cdots + C_{2k-2}C_2 + C_{2k}C_0$. On the other hand, differentiate (8) term by term to obtain:

$$\frac{1}{e^x-1} - \frac{x}{(e^x-1)^2} - \frac{x}{e^x-1} = -\frac{1}{2} + \sum_{k\geq 1} 2kC_{2k}\, x^{2k-1}$$

Multiply by x and use (8) to obtain:

$$\left(-\frac{x}{2} + \sum_{k\geq 0} C_{2k}x^{2k}\right) - \frac{x^2}{(e^x-1)^2} - x\left(-\frac{x}{2} + \sum_{k\geq 0} C_{2k}x^{2k}\right) = -\frac{x}{2} + \sum_{k\geq 1} 2kC_{2k}x^{2k}$$

Hence the coefficient of x^{2k} for $k \geq 2$ in the expansion of $x^2/(e^x-1)^2$ is $(1-2k)C_{2k}$. Therefore

$$(1-2k)C_{2k} = C_0C_{2k} + C_2C_{2k-2} + \cdots + C_{2k-2}C_2 + C_{2k}C_0$$

and, since $C_0 = 1$, we get the required relation. □

In terms of Bernoulli numbers, the convolution formula of Lemma 4.1 reads as follows:

$$-(2k+1)B_{2k} = \sum_{i=1}^{k-1} \binom{2k}{2i} B_{2i}B_{2k-2i}$$

This recurrence provides yet another way of proving that Bernoulli numbers are rational numbers.

Lemma 4.1 and the convolution relation (7), together with the fact that $\zeta(2) = \pi^2/6$, yield $\zeta(2k) = (-1)^{k+1}C_{2k}(2\pi)^{2k}/2$. But this is just (6).

Finally, formula (6) can be derived from the partial fraction expansion of the cotangent function:

$$\pi\cot(\pi x) = \frac{1}{x} + \sum_{n\geq 1}\left(\frac{1}{x+n} + \frac{1}{x-n}\right) \qquad (x \in \mathbb{R} \setminus \mathbb{Z})$$

For $0 < |x| < 1$ we can write:

$$(\pi x)\cot(\pi x) = 1 + 2\sum_{n\geq 1}\frac{x^2}{x^2 - n^2} = 1 - 2\sum_{n\geq 1}\frac{(x/n)^2}{1-(x/n)^2}$$
$$= 1 - 2\sum_{n\geq 1}\Big(\sum_{k\geq 1}(x/n)^{2k}\Big) = 1 - 2\sum_{k\geq 1}\zeta(2k)x^{2k}$$

On the other hand we have:

$$(\pi x)\cot(\pi x) = (\pi x) i\frac{e^{i(\pi x)} + e^{-i(\pi x)}}{e^{i(\pi x)} - e^{-i(\pi x)}} = (\pi i x)\frac{e^{2\pi i x}+1}{e^{2\pi i x}-1} = \pi i x + \frac{2\pi i x}{e^{2\pi i x}-1}$$

Putting $z = 2\pi i x$ in

$$\frac{z}{e^z - 1} = -\frac{z}{2} + \sum_{k\geq 0} B_{2k}\frac{z^{2k}}{(2k)!}$$

we are lead to

$$(\pi x)\cot(\pi x) = 1 + \sum_{k\geq 1}\frac{(2\pi i)^{2k}B_{2k}}{(2k)!}x^{2k}.$$

We obtain (6) by equating the coefficients of x^{2k} in the two Taylor expansions for $(\pi x)\cot(\pi x)$. This is the method explained in [1, Ch.20]; a related discussion is carried out in [9, pp.196–197].

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