# The Kepler Conjecture

## And Its Formalization

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# What is a Formal Proof?

- Re-writing proofs such that a computer can read it
- Breaking down logical inferences to their basic mathematical axioms

# Computer Aided Proofs

- Quadratic Reciprocity
  - Proof used: Godel
  - Formalized by Shankar

# Computer Aided Proofs, cont.

- Fundamental Theorem of Calculus
  - Proof used: Eisenstein
  - Formalized by Harrison

# Computer Aided Proofs, cont.

- Fundamental Theorem of Algebra
  - Proof used: Brynski
  - Formalized by Milewski

# Computer Aided Proofs, cont.

- Four-Color Theorem
  - Proof used: Robertson
  - Formalized by Gonthier

# Computer Aided Proofs, cont.

- Jordan Curve Theorem
  - Proof used: Thomassen
  - Formalized by Hales

**The Formal Jordan Curve Theorem**

$$
\begin{aligned}
&\forall C.\ \mathit{simple\_closed\_curve}\ \mathit{top2}\ C \ \Rightarrow \\
&\quad (\ \exists A\, B.\ \mathit{top2}\ A \wedge \mathit{top2}\ B\ \wedge \\
&\qquad \mathit{connected}\ \mathit{top2}\ A \wedge \mathit{connected}\ \mathit{top2}\ B\ \wedge \\
&\qquad A \neq \emptyset \wedge B \neq \emptyset \wedge \\
&\qquad A \cap B = \emptyset \wedge A \cap C = \emptyset \wedge B \cap C = \emptyset \wedge \\
&\qquad A \cup B \cup C = \mathit{euclid}\ 2\ )
\end{aligned}
$$

HOL Light Code

# Computer Aided Proofs, cont.

- Brouwer Fixed Point Theorem
  - Proof used: Kuhn
  - Formalized by Harrison

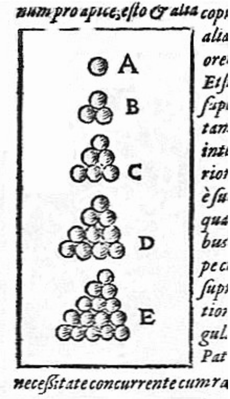
# Computer Aided Proofs, cont.

- Cauchy Residue
  - A classical Proof
  - Formalized by Harrison

# Early History

Johannes Kepler (1606)
Conception

Carl Friedrich Gauss (1831)
Proof of Regular Lattices



# 1900's

László Fejes Tóth (1953) Finite List of Packings

Claude Ambrose Rogers (1958) Upper Limit of Packing Density

Wu-Yi Hsiang (1990) Geometric Proof (Invalid)

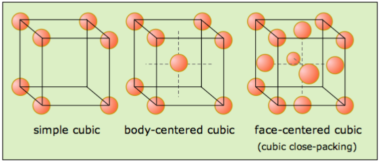
Thomas Hales (1992) Linear Programming

# The Goal

The main goal in proving Kepler's Conjecture is showing that the Cubic Close Facing, and Hexagonal facing, have the same, neither better nor worse, average packing density than any other configuration.

# The Proof

- Claude Ambrose Rogers defines Upper Limit
  - Done by expanding units of densest configuration in $\mathbb{R}$,$\mathbb{R}^2$, $\mathbb{R}^3$

# Hales

- Hales found the answer in minimizing a formula

# Hales

- Hales found the answer in minimizing a formula
- He then did most minimization's by hand

# Not Good Enough

- After four years of deliberation, it was found to be incomplete
- Hales then set off in 2003 to start FlySpeck

# Formalizing

- The main differences between the original proof and the formal proof
    - Topological results concerning plane graphs were replaced with combinatorial results from hypermaps
    - Formal proof used a different partition of geometric space
    - Formal proof added precision to explicitly state all hypotheses

# Formalizing

- The formal proof focused on key concepts, namely:
  - Spherical Trigonometry
  - Volume
  - Hypermaps
  - Fan
  - Polyhedrals
  - Voronoi Partitions
  - Linear Programming
  - Nonlinear Inequalities of Trigonometric Functions

# Proof Details

- Density:
  - Defined as the limit of the density obtained within finite containers
  - For simplicity, this container is a sphere
- An error variable is also defined within this density

# Publication

In 2017 the paper "A Formal Proof of Kepler Conjecture" was accepted and published.

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