GROUP QUASIMORPHISMS

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1. Quasimorphisms: definition and examples

Let Γ be a group. A map $\phi : \Gamma \to \mathbb{R}$ is a *quasimorphism* if it satisfies

$$\sup_{g,h\in\Gamma} |\phi(g) + \phi(h) - \phi(gh)| < \infty,$$

in which case one defines the *defect* $D_\phi := \sup_{g,h\in\Gamma} |\phi(g) + \phi(h) - \phi(gh)|$.

A bounded perturbation of a quasimorphism of Γ is again a quasimorphisms of Γ. In order to eliminate this "bounded noise", we consider the following relation of *asymptotic equivalence* on the quasimorphisms of Γ:

$$\phi \sim \phi' \text{ iff } \sup_{g\in\Gamma} |\phi(g) - \phi'(g)| < \infty$$

We denote by $X(\Gamma)$ the (real) vector space of quasimorphisms modulo asymptotic equivalence.

It turns out (see the next Lemma) that each asymptotic equivalence class has a natural representative, namely the unique quasimorphism with the property that it is an actual morphism on every cyclic subgroup. Let us make a proper definition as follows. A *quasicharacter* is a quasimorphism $\phi : \Gamma \to \mathbb{R}$ satisfying $\phi(g^n) = n\phi(g)$ for all $g \in \Gamma$, $n \in \mathbb{Z}$. Note that a quasicharacter ϕ is constant on conjugacy classes:

$$|\phi(x^{-1}gx) - \phi(g)| = \frac{|\phi(x^{-1}g^n x) - \phi(g^n)|}{n} \le \frac{2D_\phi}{n}$$

and we get $\phi(x^{-1}gx) = \phi(g)$ by letting $n \to \infty$.

Lemma 1.1. *Every asymptotic equivalence class in $X(\Gamma)$ contains a unique quasicharacter.*

Proof. We start by showing the uniqueness part. Let ϕ, ϕ' be asymptotically equivalent quasicharacters. For each $g \in \Gamma$ we have $|\phi(g) - \phi'(g)| = \frac{1}{n}|\phi(g^n) - \phi'(g^n)|$. Since $|\phi(g^n) - \phi'(g^n)|$ is bounded as $n \to \infty$, it follows that $\phi(g) = \phi'(g)$.

Next, we show the existence part. Let ϕ be a quasimorphism and put

$$\overline{\phi}(g) = \lim_{n\to\infty} \frac{\phi(g^n)}{n}.$$

First, we claim that $\overline{\phi}(g)$ is well-defined. Recall that, for a non-negative sequence (a_n) satisfying $a_{m+n} \le a_m + a_n$, the limit $\lim_{n\to\infty} a_n/n$ is well-defined. From

$$\phi(g^{m+n}) \le \phi(g^m) + \phi(g^n) + D_\phi$$

we have that the sequence with general term $b_n = \phi(g^n) + D_\phi$ is subadditive. We modify b_n by a linear term so as to guarantee non-negative values. An obvious induction yields

$$|\phi(g^n) - n\phi(g)| \le (n-1)D_\phi$$

making $a_n = b_n - n\big(\phi(g) - D_\phi\big)$ non-negative and subadditive. Since $\lim_{n\to\infty} a_n/n$ is well-defined, it follows that $\overline{\phi}(g)$ is well-defined.

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Second, we show that $\overline{\phi}$ is a quasicharacter. First one gets a bound on the price paid for interchanging two group elements:

$$|\phi(xghy) - \phi(xhgy)| \leq 6D_\phi$$

It follows that

$$|\phi((gh)^n) - \phi(g^n h^n)| \leq 6(n-1)D_\phi$$

hence

$$|\phi(g^n) + \phi(h^n) - \phi((gh)^n)| \leq D_\phi + |\phi(g^n h^n) - \phi((gh)^n)| \leq 6nD_\phi.$$

Thus $|\overline{\phi}(g) + \overline{\phi}(h) - \overline{\phi}(gh)| \leq 6D_\phi$. This proves that $\overline{\phi}$ is a quasimorphism. We still have to show that $\overline{\phi}(g^k) = k\overline{\phi}(g)$ for all integers k. For $k \geq 0$, this is clear from the definition of $\overline{\phi}$. Therefore, it suffices to check $\overline{\phi}(g^{-1}) = -\overline{\phi}(g)$; this follows immediately from $|\phi(g^n)+\phi(g^{-n})| \leq D_\phi + |\phi(1)|$.

Finally, $\overline{\phi}$ is equivalent to ϕ: $|\phi(g^n) - n\phi(g)| \leq (n-1)D_\phi$ yields $|\overline{\phi}(g) - \phi(g)| \leq D_\phi$. ☐

Let $\chi(\Gamma)$ be the (real) vector space of characters of Γ, i.e., morphisms $\Gamma \to \mathbb{R}$. Identifying $X(\Gamma)$ with the space of quasicharacters, we have that $\chi(\Gamma)$ is a subspace of $X(\Gamma)$. We view the (real) vector space

$$Q(\Gamma) := X(\Gamma)/\chi(\Gamma)$$

as a measure for the "non-triviality" of quasimorphisms on Γ.

In the following propositions, we compute $Q(\Gamma)$ for some groups Γ.

Proposition 1.2. *If Γ is amenable then $Q(\Gamma) = 0$.*

Proof. We show that every quasimorphism $\phi : \Gamma \to \mathbb{R}$ is asymptotic to a morphism. Since the real-valued map on Γ given by $x \mapsto \phi(gx) - \phi(x)$ is bounded, we may define $\overline{\phi} : \Gamma \to \mathbb{R}$ as follows:

$$\overline{\phi}(g) = \int_\Gamma \big(\phi(gx) - \phi(x)\big)\mathrm{d}x$$

Then $\overline{\phi}$ is a morphism. Indeed:

$$\begin{aligned} \overline{\phi}(gh) &= \int_\Gamma \big(\phi(ghx) - \phi(x)\big)\mathrm{d}x \\ &= \int_\Gamma \big(\phi(ghx) - \phi(hx)\big)\mathrm{d}x + \int_\Gamma \big(\phi(hx) - \phi(x)\big)\mathrm{d}x = \overline{\phi}(g) + \overline{\phi}(h) \end{aligned}$$

Furthermore, we have

$$|\overline{\phi}(g) - \phi(g)| = \Big|\int_\Gamma \big(\phi(gx) - \phi(x) - \phi(g)\big)\mathrm{d}x\Big| \leq \int_\Gamma \big|\phi(gx) - \phi(x) - \phi(g)\big|\mathrm{d}x \leq D_\phi$$

which shows that ϕ and $\overline{\phi}$ are asymptotically equivalent. ☐

Proposition 1.3. *If Γ is boundedly generated then $Q(\Gamma)$ is finite dimensional.*

Proof. Let $S = \{s_1, \dots, s_n\} \subseteq \Gamma$ be a set that boundedly generates Γ. We claim that the linear map from the space of quasicharacters $X(\Gamma)$ to $\mathbb{R}^n$, given by $\phi \mapsto (\phi(s_1), \dots, \phi(s_n))$, is injective. It will follow that $X(\Gamma)$, hence $Q(\Gamma)$ as well, is finite dimensional.

Let ϕ be a quasicharacter with $\phi(s) = 0$ for every $s \in S$. Then $\phi(s^k) = 0$ for every $s \in S$ and every $k \in \mathbb{Z}$. By the bounded generation hypothesis, there is a positive integer N such that each element $g \in \Gamma$ can be written as $g = s_{i_1}^{k_1} \dots s_{i_N}^{k_N}$ for some $s_{i_j} \in S$ and integers k_{i_j}. Then

$$|\phi(g)| = |\phi(g) - \phi(s_{i_1}^{k_1}) - \dots - \phi(s_{i_N}^{k_N})| \leq (N-1)D_\phi$$

which shows that ϕ is bounded, hence $\phi = 0$. ☐

Proposition 1.4. *If $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, where $n \geq 3$, then $Q(\Gamma) = 0$.*

Proof. We show that $X(\Gamma) = 0$, i.e., that every quasimorphism on Γ vanishes identically. In general, a quasicharacter is bounded on commutators: from

$$\phi(ghg^{-1}h^{-1}) = \phi(ghg^{-1}h^{-1}) - \phi(h) - \phi(h^{-1}) = \phi(ghg^{-1}h^{-1}) - \phi(ghg^{-1}) - \phi(h^{-1})$$

one sees that $|\phi(ghg^{-1}h^{-1})| \leq D_\phi$ for any group elements g, h. When $n \geq 3$, every elementary matrix in $\mathrm{SL}_n(\mathbb{Z})$ is a commutator of elementary matrices, so every quasicharacter on $\mathrm{SL}_n(\mathbb{Z})$ is bounded on elementary matrices. As $\mathrm{SL}_n(\mathbb{Z})$ is boundedly generated by elementary matrices, it follows that every quasicharacter on $\mathrm{SL}_n(\mathbb{Z})$ is bounded, hence vanishing. □

Proposition 1.5 (Brooks)**.** *If $\Gamma = F_2$ then $Q(\Gamma)$ is infinite dimensional.*

Many more results in this vein are currently known. Let us record here only one such result, due to Epstein and Fujiwara [1]: if Γ is a non-elementary hyperbolic group then $Q(\Gamma)$ is infinite dimensional.

Proof. Let a, b be the generators of F_2. In what follows, words are assumed to be reduced.

For a non-trivial word w, let $\#w(x)$ denote the number of appearances of w in x. Define

$$\phi_w(x) = \#w(x) - \#w^{-1}(x).$$

Note that ϕ_a is the morphism $a \mapsto 1, b \mapsto 0$ and ϕ_b is the morphism $a \mapsto 0, b \mapsto 1$, and that they form a basis for $\chi(\Gamma)$.

We claim that ϕ_w is a quasimorphism. If there is no cancelation in the product xy, then

$$\#w(x) + \#w(y) \leq \#w(xy) \leq \#w(x) + \#w(y) + |w| - 1$$

and consequently

$$\#w^{-1}(x) + \#w^{-1}(y) \leq \#w^{-1}(xy) \leq \#w^{-1}(x) + \#w^{-1}(y) + |w| - 1$$

which together yield

$$|\phi_w(xy) - \phi_w(x) - \phi_w(y)| \leq |w| - 1.$$

In general, there is a subword z that is canceled in the product xy. Write $x = x'z$, $y = z^{-1}y'$, and bound

$$|\phi_w(xy) - \phi_w(x) - \phi_w(y)| = |\phi_w(x'y') - \phi_w(x'z) - \phi_w(z^{-1}y')|$$

by

$$|\phi_w(x'y') - \phi_w(x') - \phi_w(y')| + |\phi_w(x') + \phi_w(z) - \phi_w(x'z)| + |\phi_w(y') + \phi_w(z^{-1}) - \phi_w(z^{-1}y')|.$$

By the first part, the above expression is bounded by $3(|w| - 1)$. We conclude that ϕ_w is, indeed, a quasimorphism.

Next we claim that $\{\phi_{a^n b^n}\}_{n \geq 1}$ are linearly independent in $Q(\Gamma)$. Otherwise, we would have

$$\sup_{g \in \Gamma} |\phi_{a^{n+1}b^{n+1}} + c_n \phi_{a^n b^n} + \cdots + c_1 \phi_{ab} + c_a \phi_a + c_b \phi_b| < \infty$$

for some $c_n, \ldots, c_1, c_a, c_b \in \mathbb{R}$. Evaluating on a^k and b^k, where $k \geq 1$, gives $c_a = 0$ and $c_b = 0$. Evaluating on $(ab)^k$, where $k \geq 1$, gives $c_1 = 0$. And so on. Finally, evaluating on $(a^{n+1}b^{n+1})^k$, where $k \geq 1$, gives a contradiction. □

See [2] for a detailed investigation of quasimorphisms on free groups, particularly the construction of the Faiziev quasicharacters which are more natural than the Brooks quasimorphisms we considered.

2. Bounded cohomology for groups

Let V be a Banach space. The group cohomology $\mathrm{H}^*(\Gamma, V)$ arises from the complex

$$V = C^0(\Gamma, V) \to C^1(\Gamma, V) \to C^2(\Gamma, V) \to \dots$$

where $C^n(\Gamma, V) = \{\phi : \Gamma^n \to V\}$ and the differential d is given by the following formula:

$$\begin{aligned} d\phi(g_1, \dots, g_{n+1}) &= \phi(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i \phi(g_1, \dots, g_{i-1}, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} \phi(g_1, \dots, g_n) \end{aligned}$$

To get the bounded cohomology $\mathrm{H}_b^*(\Gamma, V)$, we consider the subcomplex

$$V = C_b^0(\Gamma, V) \to C_b^1(\Gamma, V) \to C_b^2(\Gamma, V) \to \dots$$

where $C_b^n(\Gamma, V) = \{\phi : \Gamma^n \to V |\ \phi \text{ is bounded}\}$ and the differential d is the same. There is a natural comparison map $\mathrm{H}_b^*(\Gamma, V) \to \mathrm{H}^*(\Gamma, V)$.

As $\mathrm{H}_b^1(\Gamma, V)$ consists of the bounded maps $\phi : \Gamma \to V$ which satisfy $\phi(gh) = \phi(g) + \phi(h)$, we have $\mathrm{H}_b^1(\Gamma, V) = 0$. We focus on the second bounded cohomology group $\mathrm{H}_b^2(\Gamma, V)$.

Proposition 2.1. *There is an isomorphism* $Q(\Gamma) \simeq \ker\big(\mathrm{H}_b^2(\Gamma, \mathbb{R}) \to \mathrm{H}^2(\Gamma, \mathbb{R})\big)$.

Proof. Start with the well-defined map $X(\Gamma) \to \mathrm{H}_b^2(\Gamma, \mathbb{R})$ given by $\phi \mapsto [d\phi]$; here ϕ actually stands for an asymptotic equivalence class. If $[d\phi] = 0$ then $d\phi = d\beta$ for some bounded $\beta : \Gamma \to \mathbb{R}$. Hence $d\phi = 0$, so $\phi \in \chi(\Gamma)$. Conversely, $\chi(\Gamma)$ is in the kernel.

We get an injective map $X(\Gamma)/\chi(\Gamma) \to \ker\big(\mathrm{H}_b^2(\Gamma, \mathbb{R}) \to \mathrm{H}^2(\Gamma, \mathbb{R})\big)$ given by $\phi + \chi_\Gamma \mapsto [d\phi]$. For surjectivity, an element in $\ker\big(\mathrm{H}_b^2(\Gamma, \mathbb{R}) \to \mathrm{H}^2(\Gamma, \mathbb{R})\big)$ is of the form $[d\phi]$ where ϕ is defined up to $\chi(\Gamma)$ and up to perturbations by bounded maps $\beta : \Gamma \to \mathbb{R}$. □

In the case of F_2, the vanishing of $\mathrm{H}^2(F_2, \mathbb{R})$ implies that $\mathrm{H}_b^2(F_2, \mathbb{R})$ is described entirely by the space $Q(F_2)$ of non-trivial quasimorphisms. In particular:

Corollary 2.2. $\mathrm{H}_b^2(F_2, \mathbb{R})$ *is infinite dimensional.*

3. Hochschild cohomology for algebras

Let A be a complex algebra and V an $A - A$ bimodule. Let $L^n(A, V)$ denote the n-linear maps from A^n to V. The Hochschild cohomology $\mathrm{H}^*(A, V)$ is given by the complex

$$V = L^0(A, V) \to L^1(A, V) \to L^2(A, V) \to \dots$$

with differential

$$\begin{aligned} d\phi(a_1, \dots, a_{n+1}) &= a_1 \phi(a_2, \dots, a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i \phi(a_1, \dots, a_{i-1}, a_i a_{i+1}, \dots, a_{n+1}) + (-1)^{n+1} \phi(a_1, \dots, a_n) a_{n+1}. \end{aligned}$$

For instance, $\mathrm{H}^1(A, V)$ consists of *derivations* (linear maps $\phi : A \to V$ satisfying $\phi(ab) = a\phi(b) + \phi(a)b$ for all $a, b \in A$) modulo *inner derivations* (linear maps $\phi : A \to V$ of the form $\phi(a) = av - va$ for some $v \in V$).

For topological algebras, Hochschild cohomology is taken with respect to a subcomplex of $L^*(A, V)$ obtained by imposing suitable continuity conditions. Let A be a Banach algebra and V a Banach $A - A$ bimodule, meaning that V is a Banach space and A acts by bounded operators on both sides. One may take $V = A$ or some Banach super-algebra, e.g. $A = \mathrm{C}^*_\mathrm{r}\Gamma$ or $\mathrm{L}\Gamma$ and $V = \mathcal{B}(\ell^2\Gamma)$. Let $\mathcal{B}^n(A, V)$ denote the bounded n-linear maps from A^n to V. The (continuous) Hochschild cohomology $\mathrm{H}^*(A, V)$ is given by the complex

$$V = \mathcal{B}^0(A, V) \to \mathcal{B}^1(A, V) \to \mathcal{B}^2(A, V) \to \dots$$

with differential as above.

In the next theorem, it is natural to consider complex-valued bounded group cohomology. The results we had for the real-valued situation remain unchanged. Recall that the group algebra $\ell^1\Gamma$ carries a trace given by the following formula:

$$\mathrm{tr}(\sum a_g g) = a_1$$

Theorem 3.1 (Johnson). $\mathrm{H}_b^*(\Gamma,\mathbb{C})$ *embeds in* $\mathrm{H}^*(\ell^1\Gamma,\ell^1\Gamma)$.

Proof. We construct chain maps $M : C_b^*(\Gamma,\mathbb{C}) \to \mathcal{B}^*(\ell^1\Gamma,\ell^1\Gamma)$ and $m : \mathcal{B}^*(\ell^1\Gamma,\ell^1\Gamma) \to C_b^*(\Gamma,\mathbb{C})$ with $mM = \mathrm{id}$. It will follow that $\mathrm{H}_b^*(\Gamma,\mathbb{C})$ embeds in $\mathrm{H}^*(\ell^1\Gamma,\ell^1\Gamma)$.

The map $M : C_b^n(\Gamma,\mathbb{C}) \to \mathcal{B}^n(\ell^1\Gamma,\ell^1\Gamma)$ is the multiplication operator $\phi \mapsto M_\phi$, where M_ϕ is determined from

$$(M_\phi)(g_1,\dots,g_n) = \phi(g_1,\dots,g_n)g_1\dots g_n$$

Then $\|M_\phi\| = \|\phi\|_\infty$, so M itself is linear and continuous of norm 1 if we equip $C_b^n(\Gamma,\mathbb{C})$ with the sup-norm and $\mathcal{B}^n(\ell^1\Gamma,\ell^1\Gamma)$ with the operator norm.

The map $m : \mathcal{B}^n(\ell^1\Gamma,\ell^1\Gamma) \to C_b^n(\Gamma,\mathbb{C})$ is defined as follows:

$$(m\Phi)(g_1,\dots,g_n) = \mathrm{tr}\big(\Phi(g_1,\dots,g_n)(g_1\dots g_n)^{-1}\big)$$

Then $|(m\Phi)(g_1,\dots,g_n)| \le \|\Phi(g_1,\dots,g_n)\| \le \|\Phi\|$, so $\|m\Phi\|_\infty \le \|\Phi\|$. Thus m is also linear and continuous.

One checks that M and m are chain maps, i.e., $dM = Md$ and $dm = md$. Furthermore:

$$mM_\phi(g_1,\dots,g_n) = \mathrm{tr}\big(M_\phi(g_1,\dots,g_n)(g_1\dots g_n)^{-1}\big) = \mathrm{tr}(\phi(g_1,\dots,g_n)1) = \phi(g_1,\dots,g_n)$$

This means that $mM = \mathrm{id}$ on $C_b^n(\Gamma,\mathbb{C})$, as desired. □

Corollary 3.2. $\mathrm{H}^2(\ell^1F_2,\ell^1F_2)$ *is infinite dimensional.*

As hinted by the generic vanishing of $\mathrm{H}_b^1(\Gamma,\mathbb{C})$, it turns out that $\mathrm{H}^1(\ell^1\Gamma,\ell^1\Gamma) = 0$ for every group Γ (due to Johnson and Ringrose). For group von Neumann algebras, there is a similar vanishing result: $\mathrm{H}^1(\mathrm{L}\Gamma,\mathrm{L}\Gamma) = 0$, a particular case of a theorem due to Kadison and Sakai. The computation of $\mathrm{H}^2(\mathrm{L}F_2,\mathrm{L}F_2)$ is still an open problem, though a vanishing result is expected.

We end with a problem raised in [5]:

> [Problem 8.3.4] How is the bounded cohomology of Γ related to the Hochschild cohomology of $\mathrm{C}_\mathrm{r}^*\Gamma$, or to the Hochschild cohomology of $\mathrm{L}\Gamma$?

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