A First Look at Circle Packing in the Plane

Connor Kempf

IUPUI

April 2022

- Given a certain radius, how densely can we pack circles of that radius in the plane?
- It was long conjectured that the densest possible packing was the hexagonal packing
- The hexagonal packing achieves a density of $\frac{\pi}{2\sqrt{3}}$
- This conjecture was proven by Gauss.
- This presentation will offer a solution to this problem.

- What is a lattice in $\mathbb{R}^2$?
- Take two linearly independent vectors x_1, x_2 in $\mathbb{R}^2$.
- The set of all integer combinations of these vectors is a lattice.
- That is, $\{ax_1 + bx_2 : a, b \in \mathbb{Z}\}$ is the corresponding lattice.

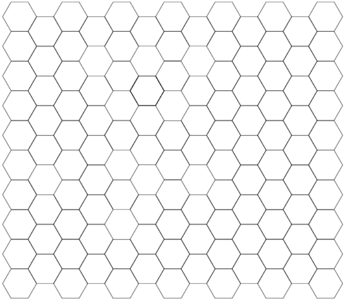

Figure: *Camia and Newman. 'Portion of the Hexagonal Lattice.'*

- B is a matrix with column vectors x_1, x_2.
- $\mathrm{L} = \{ax_1 + bx_2 : a, b \in \mathbb{Z}\}$
- $\det(\mathrm{L}) := |\det(\mathrm{B})|$
- $\det(\mathrm{L})$ is the area of the parallelogram with sides x_1 and x_2.

A **Voronoi cell** in a lattice L is described by the set of points
$\{x \in \mathbb{R}^2 : ||x|| \leq ||x-y|| \forall y \in \mathbb{R}^2\}$.

This is somewhat difficult to visualize simply from the definition, but Voronoi cells have relatively nice geometric constructions, as is shown below.

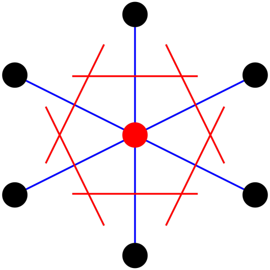

Figure: Identifying the shape of a Voronoi cell. *From Wikipedia.*

- Placing the Voronoi cell of a lattice at each lattice point will always tile the plane.
- The area of the Voronoi cell of a lattice will always be $\det(L)$.
- This can be seen by considering the basis matrix B of L as a transformation.
- Squares are stretched by a factor of $|\det(B)| = \det(L)$
- The images of $n \times n$ squares are also (approximately) covered by $n \times n$ many Voronoi cells after the transformation.

Lattices and Voronoi cells are used to define regular circle packings.

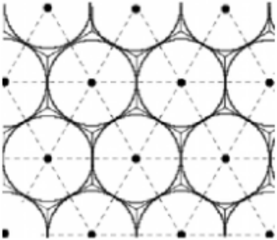

Figure: Packing circles in the Voronoi cells of the hexagonal lattice. *Fukshansky.*

- Circles are centered at lattice points
- Circles are as large as possible while still being contained in Voronoi cells

The radius r of such a circle must be $\frac{\lambda}{2}$, where λ is the least possible number such that a circle with radius λ centered at the origin contains a nonzero lattice point. *(Refer again to the figure showing how to draw a Voronoi cell.)*

The density of circles packed in a lattice L, $\Delta(\mathrm{L}) = \frac{\text{Area of a circle}}{\text{Area of a Voronoi cell}} = \frac{(\pi)r^2}{det(L)}$.

Observe that, for any lattice in which a circle packed in its Voronoi cell is not tangent to all sides of that cell, that lattice cannot have achieved maximal density.

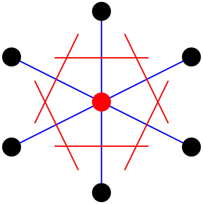

Figure: Identifying the shape of a Voronoi cell. *From Wikipedia.*

- It follows that, for a maximally dense lattice L, $||x_1|| = ||x_2|| = 2r$.
- We are free to assume that the angle θ between x_1 and x_2 will be in $(0, \frac{\pi}{2}]$.
- The area of the parallelogram bound by x_1 and x_2 will be $\det(L) = 2r \times 2r(\sin\theta)$
- So, $\Delta(L) = \frac{\pi}{4\sin(\theta)}$
- The density of our arrangement increases with decreasing (positive) θ.
- We must immediately suspect that there is some minimum possible θ.

θ must be at least $\frac{\pi}{3}$, or else there will be lattice points within a distance $\frac{\lambda}{2}$ of the origin, where λ is the minimum distance.

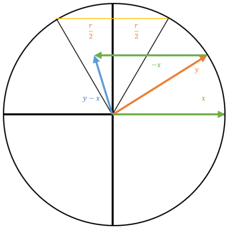

Figure: In the event that $\theta < \frac{\pi}{3}$, the difference of x_1 and x_2 produces a new lattice point, closer to the origin than allowed.

The hexagonal lattice has basis vectors $x_1 = (1, 0)$ and $x_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, the angle between which is $\frac{\pi}{3}$, so it achieves maximal density.

- We have discussed the notions of lattices and Voronoi cells
- We have used these concepts to describe regular circle packings in full generality
- Using these tools, we have proven that the hexagonal packing of circles in the plane is as dense as is possible for a regular packing!

- Camia, Frederico: "Portion of the Hexagonal Lattice." https://www.researchgate.net/figure/Portion-of-the-hexagonal-lattice_fig1_2118910
- Fukshansky, Lenny: "Revisiting the Hexagonal Lattice: On Optimal Circle Packing." *The Swiss Mathematical Society*, 2011.
- Wikipedia: https://en.wikipedia.org/wiki/Wigner%E2%80%93Seitz_cell