La Théorie Spectrale des Graphes

(Spectral (Graph Theory)), not ~~((spectral graph) theory)~~

Croix Gyurek

April 27, 2022

Matrix Definitions

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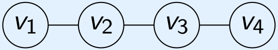
Example: 4-path



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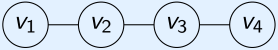

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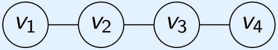

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  - Notice rows and columns of L sum to 0.

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- Spectral Theorem: both A and L have an orthonormal basis (n mutually orthogonal eigenvectors) since they're symmetric

Useful Product Identities

Both A and L play nice with dot products. For $\boldsymbol{v} \in \mathbb{R}^n$:

$$\boldsymbol{v} \cdot A\boldsymbol{v} = \sum_{(i,j)\in E} v_i v_j$$

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- $\lambda_1 = 0$; multiplicity of $0 = \#$ connected components of G
  - $L\mathbb{1} = 0$

More Classical Graph Theory

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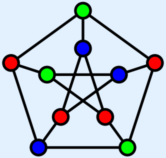

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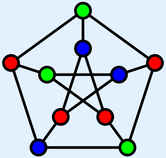

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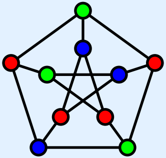

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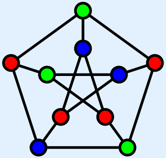

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Independence number: $\text{ind}\, G \leq n\left(1 - \frac{d_{min}}{\lambda_{max}}\right)$

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- Eigenvectors of L can give nice pictures

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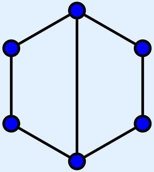
4. Use j-th coordinate of each v: vertex j goes at $(\mathrm{v}_{2,j}, \mathrm{v}_{3,j})$.

Example: 6-cycle with an extra edge

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$



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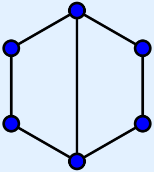

Eigenvalues: $0, 1, 2, 3, 3, 5$. Eigenvectors:
$v_2 = (-\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2})^T$ and
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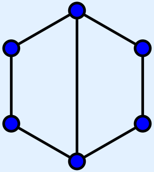

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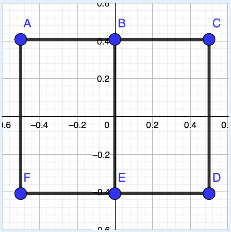

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- So let's use the coordinates of $\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$.

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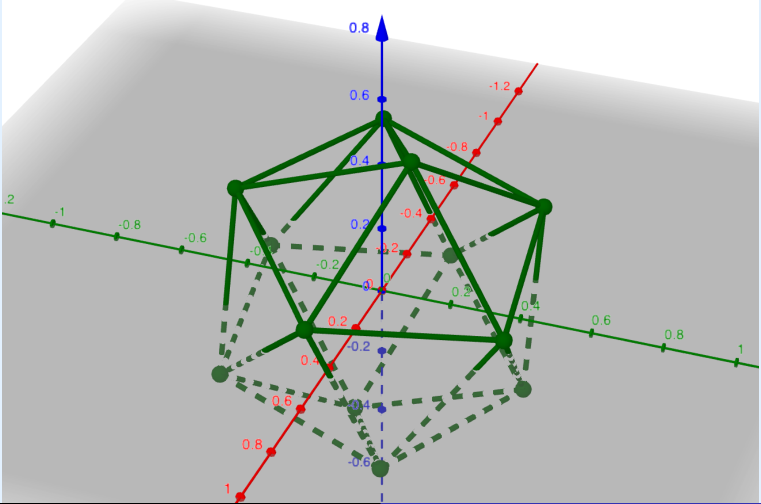

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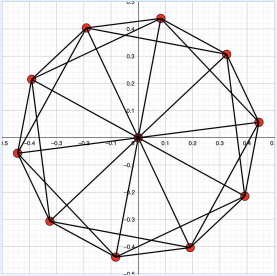

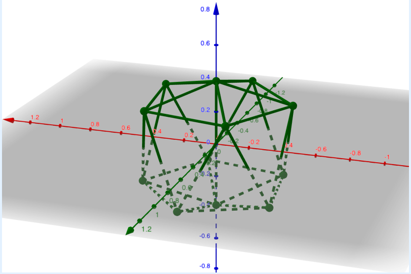

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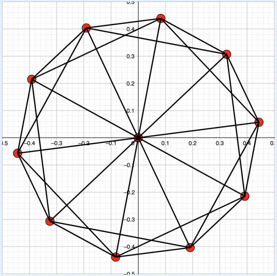

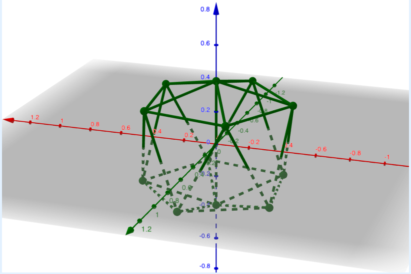

Notice the icosahedron is squished a bit!

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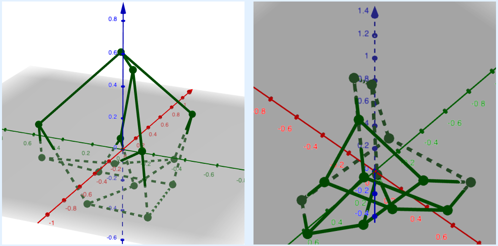

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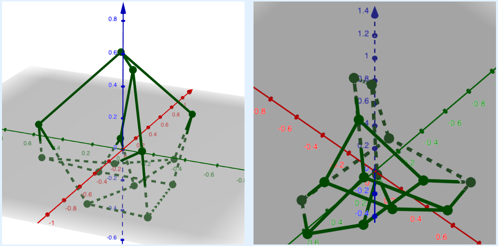
- This means the Fano plane graph is naturally six-dimensional!



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Let L be a symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then,

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The eigenvectors $\boldsymbol{v}_k$ are solutions.

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- "Adjacent vertices close" $\Rightarrow$ minimize $\sum_{(i,j)\in E} (x_i - x_j)^2$
    - Remember: this is $\boldsymbol{x} \cdot L\boldsymbol{x}$
- So we're minimizing $\boldsymbol{x} \cdot L\boldsymbol{x}$ subject to $\boldsymbol{x} \perp \mathbb{1}$ and $\|x\| = 1$.
- The solution is $\boldsymbol{x} = \mathrm{v}_2$ by CF!

Basic Idea: The One-Dimensional Case

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Similarly, requiring $\boldsymbol{x} \perp \boldsymbol{y}$ gives $\boldsymbol{y} = \boldsymbol{v}_3$, etc.

References

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