Fredholm modules and boundary actions of hyperbolic groups

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(joint work with Heath Emerson)

The boundary $\partial\Gamma$ of a non-elementary hyperbolic group Γ is a compact space on which Γ acts by homeomorphisms. In this report, we sketch the construction of certain finitely summable Fredholm modules for the crossed product C*-algebra $C(\partial\Gamma)\rtimes\Gamma$. These Fredholm modules enjoy the following features:

- homologically relevant: they represent a distinguished K-homology class, which is typically non-trivial;
- meaningful summability: roughly speaking, they are p-summable for every p greater than the Hausdorff dimension of the boundary;
- very simple form, quite unlike any other Fredholm modules known so far.

It should be noted that we are in a Type III situation - $C(\partial\Gamma)\rtimes\Gamma$ is purely infinite simple [6, 2] - so there are no finitely summable spectral triples.

The boundary extension class. The action of Γ on $\partial\Gamma$ is amenable [1]. Therefore the maximal and the reduced crossed products for the action coincide, and $C(\partial\Gamma)\rtimes\Gamma$ is a nuclear C*-algebra. For unital nuclear C*-algebras, we may identify the K^1-group of homotopy classes of odd Fredholm modules with the Ext-group of extensions by compacts. There is a natural extension of $C(\partial\Gamma)\rtimes\Gamma$ by compacts, given by the boundary compactification $\overline{\Gamma} = \Gamma \cup \partial\Gamma$:

$$0 \to \mathcal{K}(\ell^2\Gamma) \to C(\overline{\Gamma})\rtimes\Gamma \to C(\partial\Gamma)\rtimes\Gamma \to 0$$

The corresponding odd homology class, denoted $[\partial_\Gamma]$, is called the boundary extension class.

Assume that Γ is torsion-free. On the one hand, from [4] we know that there is a Poincaré duality isomorphism $\mathrm{K}^*(C(\partial\Gamma)\rtimes\Gamma) \cong \mathrm{K}_{*+1}(C(\partial\Gamma)\rtimes\Gamma)$, and that the Poincaré dual of the K^1-class $[\partial_\Gamma]$ is the K_0-class of the unit [1]. (The proof from [4] - though most likely not Poincaré duality itself - needs a mild symmetry condition on $\partial\Gamma$, but we shall disregard this minor technical point in what follows.) On the other hand, from [5] we know that the order of $[1] \in \mathrm{K}_0(C(\partial\Gamma)\rtimes\Gamma)$ is determined by the Euler characteristic of Γ as follows: [1] has finite order $|\chi(\Gamma)|$ if $\chi(\Gamma) \neq 0$, and infinite order otherwise. Combining these two facts, we obtain:

Theorem 1 (from [4] & [5]). *Let Γ be torsion-free. Then $[\partial_\Gamma]$ is non-trivial, unless $\chi(\Gamma) = \pm 1$. Furthermore, $[\partial_\Gamma]$ has infinite order if and only if $\chi(\Gamma) = 0$.*

Naive Fredholm modules for crossed products. Let us consider the general situation of a discrete group G acting by homeomorphisms on a compact space X. In order to construct a Fredholm module for the reduced crossed product $C(X)\rtimes_{\mathrm{r}} G$, we need a representation of $C(X)\rtimes_{\mathrm{r}} G$ on a Hilbert space, and a projection in that Hilbert space. For the representation, we make the obvious choice: a regular representation. If μ is a fully supported Borel probability measure on X, then $C(X)$ is faithfully represented on $L^2(X,\mu)$ by multiplication, which in turn defines a faithful representation of $C(X)\rtimes_{\mathrm{r}} G$ on $\ell^2\big(G, L^2(X,\mu)\big)$. This is the regular representation of $C(X)\rtimes_{\mathrm{r}} G$ defined by μ, and we denote it by λ_μ. Next, the choice of a projection is again the obvious one: we consider the projection of $\ell^2\big(G, L^2(X,\mu)\big)$ onto $\ell^2 G$.

In order to describe the Fredholmness and the summability of $(\lambda_\mu, P_{\ell^2 G})$, we define dynamical versions of two standard probabilistic notions, expectation and standard deviation. The *G-expectation* and the *G-deviation* of $\phi \in C(X)$ are the maps $\mathrm{E}\phi : G \to \mathbb{C}$ and $\sigma\phi : G \to [0,\infty)$ given by the formulas

$$\mathrm{E}\phi(g) = \int_X \phi \, \mathrm{d}(g_*\mu), \qquad \sigma\phi = \sqrt{\mathrm{E}|\phi|^2 - |\mathrm{E}\phi|^2}.$$

Now the Fredholmness and the summability of $(\lambda_\mu, P_{\ell^2 G})$ can be characterized by decay conditions for the G-deviation, as follows:

Proposition 2. *$(\lambda_\mu, P_{\ell^2 G})$ is a Fredholm module for $C(X)\rtimes_{\mathrm{r}} G$ if and only if $\sigma\phi \in C_0(G)$ for all $\phi \in C(X)$. Furthermore, $(\lambda_\mu, P_{\ell^2 G})$ is p-summable if and only if $\sigma\phi \in \ell^p G$ for all ϕ in a dense subalgebra of $C(X)$.*

Alternately, and interestingly, the Fredholmness of $(\lambda_\mu, P_{\ell^2 G})$ can be described by a kind of "pure proximality" à la Furstenberg:

Proposition 3. *$(\lambda_\mu, P_{\ell^2 G})$ is a Fredholm module for $C(X)\rtimes_{\mathrm{r}} G$ if and only if $g_*\mu$ only accumulates to point masses in Prob(X) as $g \to \infty$ in G.*

We need two further properties in what follows. The first is an independence result motivated by the fact that, in general, there is no canonical measure on the boundary of a hyperbolic group. The second is a multiplicativity property motivated by the desire to extend Theorem 1 to virtually torsion-free Γ. We say that two measures are *comparable* if one is between constant multiples of the other.

Proposition 4. *Let μ' be a fully supported Borel probability measure on X which is comparable to μ. Then $(\lambda_{\mu'}, P_{\ell^2 G})$ enjoys the same Fredholmness and summability as $(\lambda_\mu, P_{\ell^2 G})$. If $(\lambda_\mu, P_{\ell^2 G})$ and $(\lambda_{\mu'}, P_{\ell^2 G})$ are Fredholm modules, then they are K^1-homologous.*

Proposition 5. *Assume that $(\lambda_\mu, P_{\ell^2 G})$ is a Fredholm module for $C(X)\rtimes_{\mathrm{r}} G$, and that the measures $\{g_*\mu\}_{g\in G}$ are mutually comparable. Let $H \leq G$ be a subgroup of finite index. Then $[(\lambda_\mu, P_{\ell^2 G})] = [G:H]\cdot[(\lambda_\mu, P_{\ell^2 H})]$ in $\mathrm{K}^1(C(X)\rtimes_{\mathrm{r}} H)$.*

Back to hyperbolic groups. The boundary of a non-elementary hyperbolic group carries certain natural measures induced by "hyperbolic fillings". Namely, if Γ acts geometrically - that is, isometrically, properly, and cocompactly - on a (hyperbolic) space X, then ∂X is a topological incarnation of $\partial\Gamma$. A visual metric on ∂X is any metric comparable with $\exp(-\epsilon(\cdot,\cdot)_\bullet)$, where $(\cdot,\cdot)_\bullet$ stands for the extended Gromov product. It turns out that such metrics exist for small enough $\epsilon > 0$, and any two visual metrics are Hölder equivalent. The visual probability measures on ∂X are the normalized Hausdorff measures induced by visual metrics. Any two visual probability measures are comparable. Most importantly, visual measures are Ahlfors regular [3]: if d is a visual metric on ∂X, then the corresponding visual probability measure μ has the property that $\mu(R\text{-ball}) \asymp R^{\,\mathrm{hdim}(\partial X,d)}$. The point is that, roughly speaking, Ahlfors regularity implies that the Γ-deviation of Lipschitz maps on $(\partial X, d)$ is in $\ell^p\Gamma$ for $p > \mathrm{hdim}(\partial X, d)$. By Proposition 2, this means that $(\lambda_\mu, P_{\ell^2\Gamma})$ is a p-summable Fredholm module for $p > \mathrm{hdim}(\partial X, d)$. However, since the summability is independent of the choice of visual probability measure (Proposition 4), we are led to considering the "minimal Hausdorff dimension" of ∂X with respect to the visual metrics:

$$\mathrm{visdim}\,\partial X = \inf\{\mathrm{hdim}(\partial X, d) : d \text{ visual metric}\}.$$

We may now state our main result:

Theorem 6. *Let Γ act geometrically on X. Then, for every visual probability measure μ on ∂X, the following hold:*

i) *$(\lambda_\mu, P_{\ell^2\Gamma})$ is a Fredholm module for $C(\partial\Gamma) \rtimes \Gamma$ which is p-summable for every $p > \max\{\mathrm{visdim}\,\partial X, 2\}$. In the case when $\mathrm{visdim}\,\partial X > 2$ and it is attained, $(\lambda_\mu, P_{\ell^2\Gamma})$ is in fact $(\mathrm{visdim}\,\partial X)^+$-summable;*

ii) *$(\lambda_\mu, P_{\ell^2\Gamma})$ represents $[\partial_\Gamma]$.*

The last point is based on the fact that extending $\phi \in C(\partial\Gamma)$ by $\mathrm{E}\phi$ on Γ yields a function, denoted $\overline{\mathrm{E}}\phi$, on $\overline{\Gamma}$ which is continuous. Hence $\overline{\mathrm{E}}$ is a Γ-equivariant cp-section for $0 \to C_0(\Gamma) \to C(\overline{\Gamma}) \to C(\partial\Gamma) \to 0$, and then $\overline{\mathrm{E}}$ can be promoted to a cp-section for $0 \to \mathcal{K}(\ell^2\Gamma) \to C(\overline{\Gamma}) \rtimes \Gamma \to C(\partial\Gamma) \rtimes \Gamma \to 0$. One concludes by a Stinespring dilation argument.

From Proposition 5 we deduce a multiplicativity property for the boundary extension class: if $\Lambda \leq \Gamma$ is a subgroup of finite index, then the natural map $\mathrm{K}^1(C(\partial\Gamma) \rtimes \Gamma) \to \mathrm{K}^1(C(\partial\Lambda) \rtimes \Lambda)$ sends $[\partial_\Gamma]$ to $[\Gamma : \Lambda] \cdot [\partial_\Lambda]$. For virtually torsion-free groups, which have a well-defined notion of rational Euler characteristic, Theorem 1 and the above multiplicativity property imply the following criterion:

Corollary 7. *Let Γ be virtually torsion-free. If $\chi(\Gamma) \notin 1/\mathbb{Z}$ then $[\partial_\Gamma]$ is non-trivial. If $\chi(\Gamma) = 0$ then $[\partial_\Gamma]$ has infinite order.*

References

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