How Many Knots Are There?

It's knot so easy to show.

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Outline

1. What is Knot Theory?

2. What is a Knot?

3. How Many Knots Are There?

Knot Theory

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- Revolution: knotted DNA was discovered, and enzymes that untie that DNA.
- Knot theory is generally considered a more accessible sub-field of topology.

What is a knot?

Definition (idea)

A **knot** is a closed curve in $\mathbb{R}^3$ with no self-intersections.

Intuition is stretchy, bendy string of zero width connected at the ends with no self-intersections.

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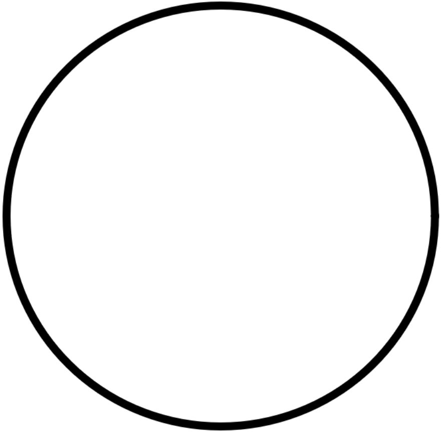
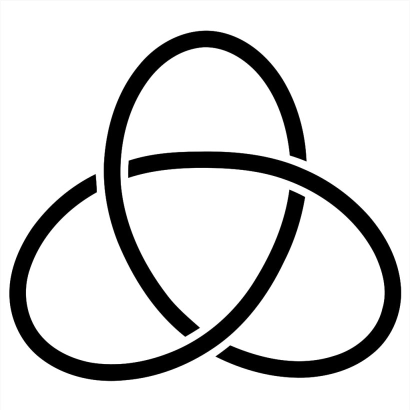
- Two knots are *equivalent* if one can be deformed to the other in a smooth, structure-preserving manner.
- That is, a deformation where the curve does not pass through itself, and does not do anything physically impossible, such as tightening portions of the curve down to a single point.

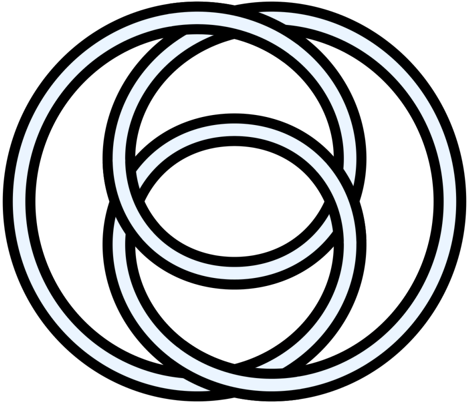
The Unknot



The Trefoil and Figure Eight

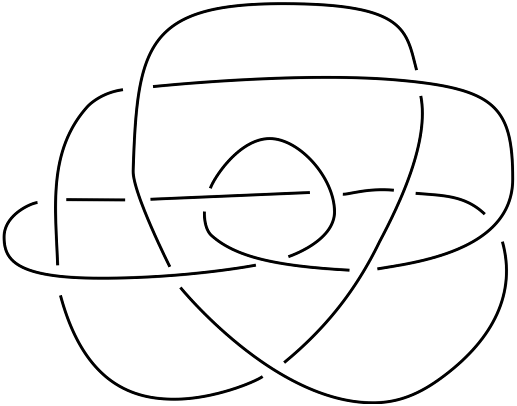


The Trefoil



The Figure-Eight

Another Knot?



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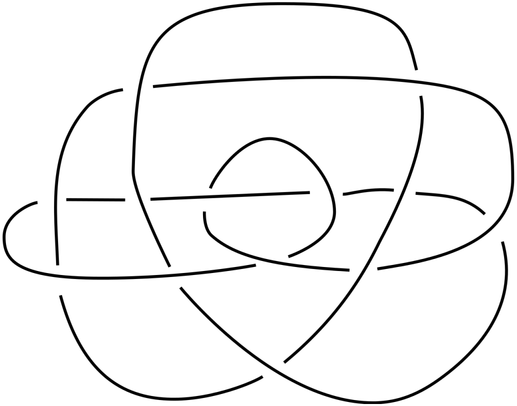

Believe it or not, this is the **Unknot**.

Knot Projections

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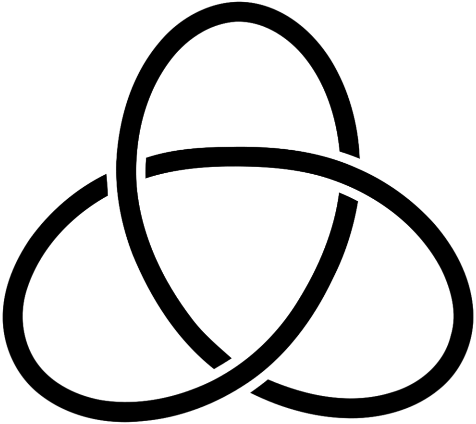
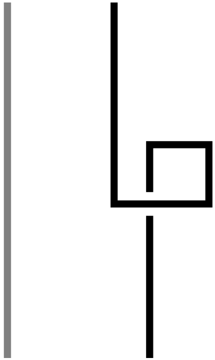


A projection of the Trefoil

Tying and Untying Knots

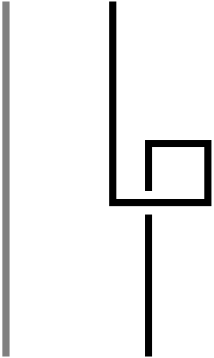
- How do we show two knots are the same?
- **Reidemeister moves** provide a formal framework for manipulating knots.
- There are **three types** (plus three mirror-images).

Reidemeister Moves



Type I

Reidemeister Moves



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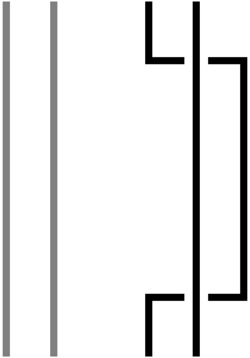
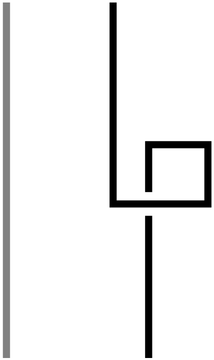


Type II

Reidemeister Moves



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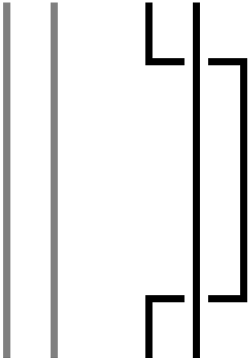
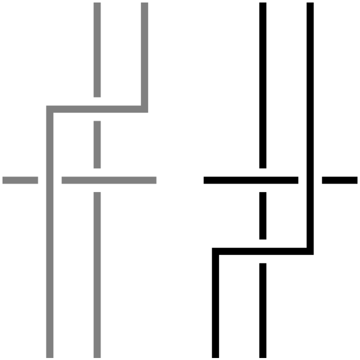


Type II



Type III

Reidemeister Moves

Theorem

There is a sequence of Reidemeister moves taking one knot projection to another if and only if the knots are equivalent.

- This theorem takes some work to prove.
- Intuitively, it is clear these do not alter the structure of the knot.

How can we tell knots apart?

- Two knot projections are equivalent if there is a sequence of Reidemeister moves between one another.
- For two knots to be different, we have to show there is *not* a sequence of Reidemeister moves between them. This is much harder.
- E.g., can we deform the trefoil into the unknot. What if it can just takes $10,000,000$ Reidemeister moves? What about $10,000,001$?
- **We need more tools to distinguish knots.**

Knot Invariants

A knot invariant is a property of a knot projection that holds for *all* projections of that knot. That is, if we have two projections of a knot, K, namely $P_1(K)$ and $P_2(K)$, then a knot invariant is a property f such that

$$f(P_1(K)) = f(P_2(K))$$

One such knot invariant is **tri-colorability**.

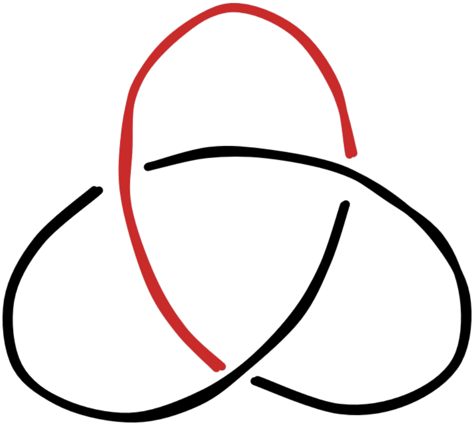
Note: This may be reminiscent of how to show that two complicated graphs are not isomorphic.

Tri-Colorability

We begin with the definition of a strand.

Definition

A **strand** is an unbroken segment in a knot projection.



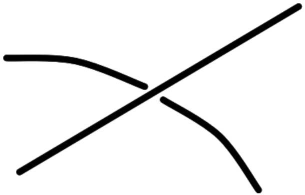
One strand in the Trefoil

Tri-Colorability

Now, we define tri-colorability...

Definition

A crossing is said to be **tri-colorable** if each strand meeting at the crossing is either the same color, or each is a different color.

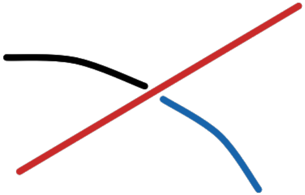


A tri-colored crossing using one color

Moreover...

Definition

A knot is tri-colorable if each crossing is tri-colorable, and all three colors are used at least once.



A tri-colored crossing using three colors

Proving tri-colorability is a knot invariant

We propose the following theorem:

Theorem

If two knot projections represent the same knot, then they are either both tri-colorable, or neither is tri-colorable.

Proof of Tri-Colorability Theorem

We know two projections represent the same knot if and only if there is a sequence of Reidemeister moves between them. What do we need to show?

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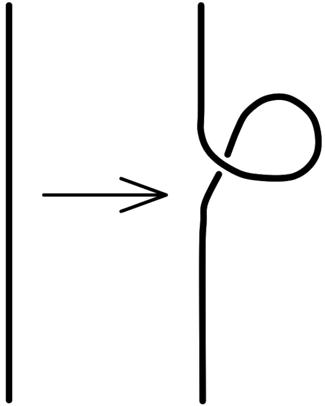
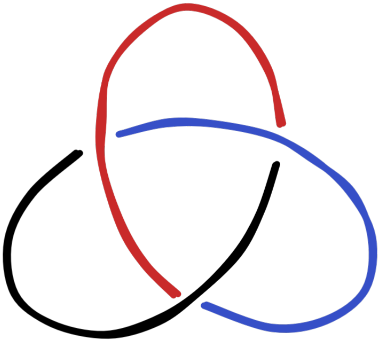
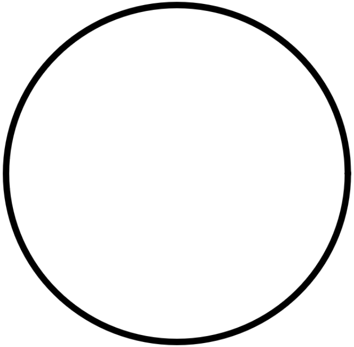


The segment remains tri-colorable

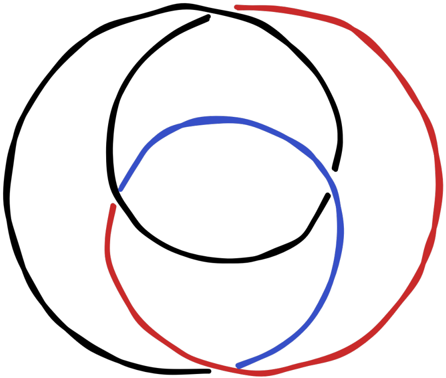
How Many Knots Are There?

At least two: the trefoil and the unknot.





More on Knots



The figure-eight is not tri-colorable.

Certainly, there are more than two distinct knots.

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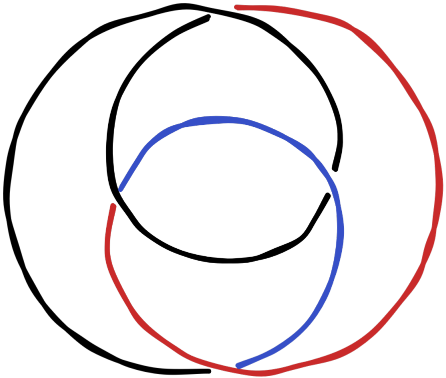

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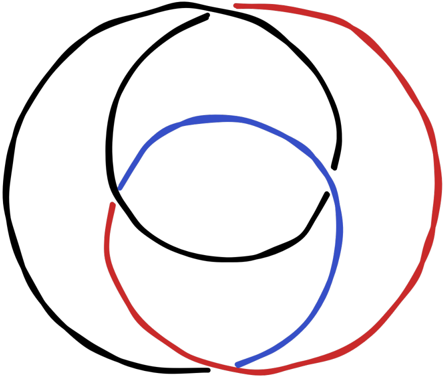

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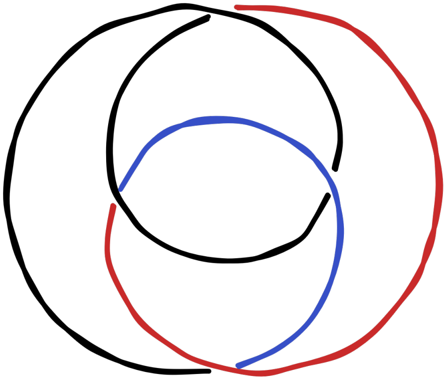

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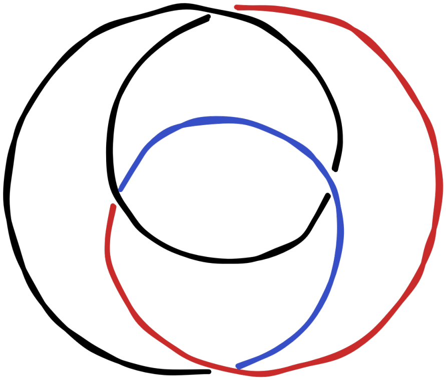

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- **Unsolved question**: prove that there are more distinct prime knots of $n + 1$ crossings than of n crossings.

The End

References

- Adams, Colin C. *The knot book. An elementary introduction to the mathematical theory of knots.* American Mathematical Society, Providence, RI, 2004. ISBN: 0-8218-3678-1
- Trefoil from `https://en.wikipedia.org/wiki/File:Trefoil_knot_left.svg`
- Figure Eight from `https://commons.wikimedia.org/wiki/File:Figure8knot-rose-limacon-curve.svg`
- Ochai Unknot from `https://commons.wikimedia.org/wiki/File:Ochiai_unknot.svg`
- Reidemeister moves from `https://en.wikipedia.org/wiki/Reidemeister_move`