The Wronskian as a Method for Introducing Vector Spaces

Daniel A. Ramras
Department of Mathematical Sciences,
New Mexico State University

http://math.nmsu.edu/∼ramras/wronskian.html

January 10, 2013

Introduction

Goal

Lead students through a process of discovery, in which they apply linear algebra to the space of differentiable functions.

The (abbreviated) worksheet you will see was designed for students in a beginning linear algebra class.

Student Background:

- *Differential calculus*
- *Basic linear algebra in Euclidean space:*
  - *Systems of equations; Vectors and matrices;*
  - *Linear dependence and linear independence;*
  - *Invertible matrices, determinants;*
  - *Linear transformations* $\mathbb{R}^n \to \mathbb{R}^m$.

Responses shown are (mostly) from students' homework.

The Vector Space of Differentiable Functions

Let $C^\infty(\mathbb{R})$ denote the set of all infinitely differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$.

Then $C^\infty(\mathbb{R})$ is a vector space, using the usual addition and scalar multiplication for functions.

Problem 1: Differentiation

Differentiation defines a *function*

$$D\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}).$$

This function D is defined by $D(f) = f'$.

(a) Calculate $D(e^x)$, $D(x^3 + x)$, $D(\cos(x))$.

$D(e^x) = e^x$, $D(x^3+x) = 3x^2+1$, $D(\cos x) = -\sin x$

(b) Show, by calculating both sides, that
$$D(2e^x + 3\cos(x)) = 2D(e^x) + 3D(\cos(x)).$$

$D(2e^x + 3\cos(x)) = 2D(e^x) + 3D(\cos(x))$.

$2e^x - 3\sin(x) = 2(e^x) + 3(-\sin(x))$

$2e^x - 3\sin(x) = 2e^x - 3\sin(x)$

(c) We've shown that D preserves the linear combination $2e^x + 3\cos(x)$. We'll now show that D is a *linear transformation* of vector spaces, meaning D preserves *all* linear combinations.

What formulas express this idea, and which rules for derivatives do these formulas correspond to?

$D(f+g) = D(f) + D(g)$ - Sum Rule

$D(cf) = cD(f)$ - Constant Rule

(d) If we compose two linear transformations, we get another linear transformation.

Example: $D^2 = D \circ D \colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is a linear transformation that sends each function f to its *second derivative.*

Calculate $D^2(2e^x + 3\cos(x))$, $D^2(e^x)$ and $D^2(\cos(x))$.

$D^2(2e^x + 3\cos x) = D(2e^x - 3\sin x) = 2e^x - 3\cos x$

$D^2(e^x) = D(e^x) = e^x$

$D^2(\cos(x)) = D(-\sin x) = -\cos x$

Compare your answers to see that

$$D^2(2e^x + 3\cos(x)) = 2D^2(e^x) + 3D^2(\cos x).$$

Problem 2: Linear Independence in $C^\infty(\mathbb{R})$:

In this problem, we'll examine whether or not certain collections of functions are *linearly independent* in the vector space $C^\infty(\mathbb{R})$.

Question: Are e^x, $\cos(x)$, and $x^3 + x$ linearly independent?

(a) What would it mean if these functions were linearly *dependent*? Write an *equation* expressing your answer.

$c_1 e^x = c_2 \cos(x) = c_3(x^3 + x)$

That would make them linearly dependent because they would be multiples of eachother

---

If $e^x, \cos(x)$ & $x^3 + x$ were linear dependent, we could say that

$c_1 e^x = c_2 \cos(x) + c_3(x^3 + x)$

$a(e^x) + b(\cos(x)) + c(x^3 + x) = 0$ where a, b, and c are not all $= 0$.

(b) Now, apply the *linear transformation* D (differentiation) to the equation from part (a) to obtain an equation relating the *derivatives* of our functions.

$aD(e^x) + bD(\cos(x)) + cD(x^3 + x) = 0$ where a, b, c not all $= 0$

Apply D again to the equation you just obtained to get an equation relating the *second* derivatives.

$D(aD(e^x) + bD(\cos(x)) + cD(x^3 + x)) = 0 \Rightarrow$

$\Rightarrow aD^2(e^x) + bD^2(\cos(x)) + cD^2(x^3 + x) = 0$ where a, b, c not all $= 0$

If e^x, $\cos(x)$, and $x^3 + x$ were linearly dependent, what would the previous equations tell you about the *columns* of the *Wronskian* matrix $W(x)$ below?

$$W(x) = \begin{bmatrix} e^x & \cos(x) & x^3 + x \\ D(e^x) & D\cos(x) & D(x^3 + x) \\ D^2(e^x) & D^2\cos(x) & D^2(x^3 + x) \end{bmatrix}$$

If e^x, $\cos(x)$, and $x^3 + x$ were linearly dependent then would use the equations from 2a.) and 2b.) showing that $W(x)\vec{c} = 0$, where $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ and $\vec{c} \neq \vec{0}$

(c) Compute

$$W(x) = \begin{bmatrix} e^x & \cos(x) & x^3 + x \\ D(e^x) & D\cos(x) & D(x^3+x) \\ D^2(e^x) & D^2\cos(x) & D^2(x^3+x) \end{bmatrix}$$

and plug in $x = 0$ to get a matrix $W(0)$.

$$W(x) = \begin{bmatrix} e^x & \cos(x) & x^3+x \\ e^x & -\sin(x) & 3x^2+1 \\ e^x & -\cos(x) & 6x \end{bmatrix}$$

$$W(0) = \begin{bmatrix} e^0 & \cos(0) & 0^3+0 \\ e^0 & -\sin(0) & 3(0)^2+1 \\ e^0 & -\cos(0) & 6(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\vec{a}_1, \vec{a}_2, \vec{a}_3$$

(d) Explain the following statement: If e^x, $\cos(x)$, and x^3+x were linearly dependent, then the columns of $W(0)$ would also be linearly dependent (in other words, $W(0)$ would fail to be invertible).

$$c_1 \begin{pmatrix} e^x \\ e^x \\ e^x \end{pmatrix} + c_2 \begin{pmatrix} \cos(x) \\ -\sin(x) \\ -\cos(x) \end{pmatrix} + c_3 \begin{pmatrix} x^3+x \\ 3x^2+1 \\ 6x \end{pmatrix} = 0 \Rightarrow c_1 \begin{pmatrix} e^0 \\ e^0 \\ e^0 \end{pmatrix} + c_2 \begin{pmatrix} \cos(0) \\ -\sin(0) \\ -\cos(0) \end{pmatrix} + c_3 \begin{pmatrix} 0^3+0 \\ 3\cdot 0^2+1 \\ 6\cdot 0 \end{pmatrix} = 0$$

$\Rightarrow c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 = 0 \Rightarrow \vec{a}_1, \vec{a}_2, \vec{a}_3$ dep. b/c c_1, c_2, c_3 not all zero

(e) Is $W(0)$ invertible? Why or why not? What does this tell you about $e^x, \cos(x)$, and $x^3 + x$?

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 1\left(0\cdot 0 - (1\cdot -1)\right) - 1\left(1\cdot 0 - 1\cdot 1\right) = 2$$

$2 \neq 0$ so $W(0)$ is invertible

and e^x $\cos(x)$ and $x^3 + x$ are linearly independent

Yes, $W(0)$ is invertible. It is because I was able to find the inverse, which is $\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & 1 & -1/2 \end{bmatrix}$. This means that e^x, $\cos(x)$, & x^3+x are linearly independent.

$W(0)$ is invertible through simply row reducing and checking if it has $n=3$ pivots.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{matrix} \\ -R1 \\ -R1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 0 \end{bmatrix} -2R2 \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

This tells us that e^x, $\cos(x)$, and x^3+x are linearly independent from each other.

Problem 3: We've seen that if $W(0)$ is invertible, our functions are independent. If $W(0)$ is singular, must our functions be *dependent*?

(a) Compute $W(0)$ for the functions $\sin(x)$, $\sin(2x)$, $\cos(x)$. Is $W(0)$ invertible?

$$W(x) = \begin{bmatrix} \sin(x) & \sin(2x) & \cos(x) \\ \cos(x) & 2\cos(2x) & -\sin(x) \\ -\sin(x) & -4\sin(2x) & -\cos(x) \end{bmatrix}$$

$$W(0) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad |W(0)| = 0$$

$W(x)$ not invertible @ $x = 0$

(b) Is $W(\pi/4)$ invertible? What can you say about independence or dependence of these functions?

$$W(\pi/4) = \begin{bmatrix} \sqrt{2}/2 & 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & -4 & -\sqrt{2}/2 \end{bmatrix}$$

$$|W(\pi/4)| = -3 \qquad W(x) \text{ is invertible at } x = \pi/4$$

Wronskin can be used to show linear independence, but not necessarily to prove linear dependence.

Concluding Remarks

For proving linear independence, the matrix

$$\begin{bmatrix} f(x_1) & g(x_1) & h(x_1) \\ f(x_2) & g(x_2) & h(x_2) \\ f(x_3) & g(x_3) & h(x_3) \end{bmatrix}$$

($x_1, x_2, x_3 \in \mathbb{R}$ distinct) is often just as useful as the Wronskian.

Pros and Cons

- *The above matrix does not involve derivatives, and does not require/reinforce the notion of linear transformation.*
- *Often* $\det W(0) \neq 0$ *can be checked without a calculator.*
- *The Wronskian has deeper connections to differential equations (Variation of Parameters).*

http://math.nmsu.edu/∼ramras/wronskian.html