Math 480

The Vector Space of Differentiable Functions

The vector space of differentiable functions. Let $C^\infty(\mathbb{R})$ denote the set of all infinitely differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$. Then $C^\infty(\mathbb{R})$ is a vector space, using the usual notions of addition and scalar multiplication for functions. For instance, if f is the function $f(x) = e^x$, and g is the function $g(x) = \sin(x)$, then $2f$ is the function $2f(x) = 2e^x$ and $f + g$ is the function $(f + g)(x) = e^x + \sin(x)$.

Problem 1: Differentiation.

Differentiation defines a *function* $D\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$. This function D is defined by $D(f) = f'$.

(a) Calculate $D(e^x)$, $D(x^3 + x)$, $D(\cos(x))$.

(b) Show, by calculating both sides, that $D(2e^x + 3\cos(x)) = 2D(e^x) + 3D(\cos(x))$.

The computation in part b) shows that the function D preserves the linear combination $2e^x + 3\cos(x)$. We'll now show that D is a *linear transformation* of vector spaces, meaning it preserves *all* such linear combinations.

(c) Let f and g be functions in $C^\infty(\mathbb{R})$. Which rule for derivatives allows you to compute $D(f+g)$? Find a formula for $D(f + g)$ in terms of $D(f)$ and $D(g)$.

(d) Let f be a function in $C^\infty(\mathbb{R})$, and let $c \in \mathbb{R}$ be a scalar. Which rule for derivatives allows you to compute $D(cf)$? Find a formula for $D(cf)$ in terms of c and $D(f)$.

Compare the formulas above to the rules for a linear transformation: you've just proven that the function $D\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is a linear transformation from the vector space $C^\infty(\mathbb{R})$ to itself!

(e) If we compose two linear transformations, we get another linear transformation (we saw this for linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$ and $S\colon \mathbb{R}^m \to \mathbb{R}^p$, but it's true in general for the same reason).

Example: $D \circ D\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is a linear transformation. Sometimes we write D^2 instead of $D \circ D$. Note that D^2 sends each function f to its *second derivative.*

Calculate $D^2(2e^x + 3\cos(x))$, $D^2(e^x)$ and $D^2(\cos(x))$.

Compare your answers above to see that

$$D^2(2e^x + 3\cos(x)) = 2D^2(e^x) + 3D^2(\cos x).$$

This example shows that D^2 preserves this particular linear combination; the fact that D^2 is a linear transformation means that it actually preserves *all* such linear combinations.

Problem 2: Linear Independence in $C^\infty(\mathbb{R})$:

In this problem, we'll examine whether or not certain collections of functions are *linearly independent* in the vector space $C^\infty(\mathbb{R})$.

Question: Are e^x, $\cos(x)$, and $x^3 + x$ linearly independent?

(a) What would it mean if these functions were linearly *dependent*? Write an *equation* expressing your answer (your equation should involve three unknown scalars, c_1, c_1, and c_3, say).

(b) Now, consider what happens if we apply the *linear transformation* D (differentiation) to the equation from part (a). Apply D to this equation to obtain an equation relating the *derivatives* of our functions (it's better not to compute those derivatives quite yet; just write $\frac{d}{dx}e^x$ or $D(e^x)$).

Apply D again to the equation you just obtained to get an equation relating the *second* derivatives.

If e^x, $\cos(x)$, and $x^3 + x$ are linearly dependent, what do the previous two equations tell you about the *columns* of the matrix below? (This matrix is called the *Wronskian*.)

$$W(x) = \begin{bmatrix} e^x & \cos(x) & x^3 + x \\ D(e^x) & D\cos(x) & D(x^3 + x) \\ D^2(e^x) & D^2\cos(x) & D^2(x^3 + x) \end{bmatrix}$$

(c) Compute the matrix $W(x) = \begin{bmatrix} e^x & \cos(x) & x^3 + x \\ D(e^x) & D\cos(x) & D(x^3+x) \\ D^2(e^x) & D^2\cos(x) & D^2(x^3+x) \end{bmatrix}$ using the derivatives you calculated in 1a), and plug in $x = 0$ to get a matrix $W(0)$.

(d) Explain the following statement: If e^x, $\cos(x)$, and x^3+x were linearly dependent, then the columns $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ of the matrix $W(0)$ that you computed above would also be linearly dependent (in other words, $W(0)$ would fail to be invertible).

(e) Is $W(0)$ invertible? Why or why not? What does this tell you about $e^x, \cos(x)$, and $x^3 + x$?

Problem 3: Use an argument similar to that above to show that the functions e^x, e^{-x}, and e^{2x} are linearly independent.

Problem 4: In the above discussion, we've seen that the Wronskian can be used to prove that certain collections of functions are linearly independent. In this problem, we'll consider whether or not the Wronskian can be used to prove that collections of functions are *dependent*.

(a) Compute $W(0)$ for the functions $\sin(x)$, $\sin(2x)$, $\cos(x)$. Is $W(0)$ invertible? What do you think this says about linear independence or dependence of these functions?

(b) Compute $W(\pi/4)$ for the same functions ($\sin(x)$, $\sin(2x)$, and $\cos(x)$). Is $W(\pi/4)$ invertible? What can you conclude about independence or dependence of these functions?