# The Atiyah - Hirzebruch Spectral Sequence

Let $X$ be a CW complex. $\forall\, k$ there is a LES in K-theory associated with the pair $(X^{(k)}, X^{(k-1)})$.

$$\tilde{K}^0(X^{(k)}/X^{(k-1)}) \xrightarrow{j^*} \tilde{K}^0(X^{(k)}) \xrightarrow{i^*} \tilde{K}^0(X^{(k-1)}) \xrightarrow{\delta} \tilde{K}^1(X^{(k)}/X^{(k-1)}) \to \cdots$$

(With the isomorphisms from Bott periodicity, this could also be considered as a six-term "exact hexagon.") Piecing together these sequences, we get the following "staircase diagram."

$$\begin{tikzcd}
 & & \arrow[d] & \\
\text{LES for } k=p+1 & & & \\
{} \arrow[r] & \tilde{K}^n(X^{(p+1)}/X^{(p)}) \arrow[r, "j_1^*"] & \tilde{K}^n(X^{(p+1)}) \arrow[r, "\delta_1"] \arrow[d, "i_1^*"] & \tilde{K}^{n+1}(X^{(p+2)}/X^{(p+1)}) \arrow[r] & {} \\
\text{LES for } k=p & & & \\
{} \arrow[r] & \tilde{K}^n(X^{(p)}/X^{(p-1)}) \arrow[r, "j_1^*"] & \tilde{K}^n(X^{(p)}) \arrow[r, "\delta_1"] \arrow[d, "i_1^*"] & \tilde{K}^{n+1}(X^{(p+1)}/X^{(p)}) \arrow[r] & {} \\
{} \arrow[r] & \tilde{K}^n(X^{(p-1)}/X^{(p-2)}) \arrow[r, "j_1^*"] & \tilde{K}^n(X^{(p-1)}) \arrow[r, "\delta_1"] \arrow[d] & \tilde{K}^{n+1}(X^{(p)}/X^{(p-1)}) \arrow[r] & {} \\
 & & {} & 
\end{tikzcd}$$

This diagram is what we call the first "page" of the spectral sequence, which is why maps have subscript 1.

Define $d_1 = \delta_1 j_1^* : \tilde{K}^n(X^{(p)}/X^{(p-1)}) \longrightarrow \tilde{K}^{n+1}(X^{(p+1)}/X^{(p)})$.

$d_1$ is a differential, i.e. $d_1^2 = 0$, because $\delta_1 j_1^* \delta_1 j_1^* = \delta_1 (0) j_1^* = 0$.

Define $A_1^{n,p} = \tilde{K}^n(X^{(p)})$ and $E_1^{n,p} = \tilde{K}^{n+1}(X^{(p+1)}/X^{(p)})$.

Now define $A_2^{n,p}$ to be $i_1^*(A_1^{n,p})$ and $E_2^{n,p}$ to be $H(E_1^{n,p}) = \operatorname{Ker} d_1 / \operatorname{Im} d_1$ at $E_1^{n,p}$. The following diagram relates the $A_2$ and $E_2$ terms:

$$\begin{tikzcd}
 & & \arrow[d] & \\
{} \arrow[r] & H(\tilde{K}^n(X^{(p)}/X^{(p-1)})) \arrow[r, "j_2^*"] & i_1^*(\tilde{K}^n(X^{(p)})) \arrow[d, "i_2^*"] \arrow[r] & {} \quad H(\tilde{K}^{n+1}(X^{(p+1)}/X^{(p)})) \\
{} \arrow[r] & H(\tilde{K}^n(X^{(p-1)}/X^{(p-2)})) \arrow[r, "j_2^*"] & i_1^*(\tilde{K}^n(X^{(p-1)})) \arrow[d, "i_2^*"] \arrow[ur, "\delta_2"] & H(\tilde{K}^{n+1}(X^{(p)}/X^{(p-1)})) \\
{} \arrow[r] & H(\tilde{K}^n(X^{(p-2)}/X^{(p-3)})) \arrow[r, "j_2^*"] & i_1^*(\tilde{K}^n(X^{(p-2)})) \arrow[d] \arrow[ur, "\delta_2"] & H(\tilde{K}^{n+1}(X^{(p-1)}/X^{(p-2)})) \\
 & & {} & {} \arrow[u]
\end{tikzcd}$$

In this diagram, $i_2^*$ is the restriction of $i_1^*$, $j_2^*$ is given by $[e] \mapsto je$, and $\delta_2$ is given by $i_1^* a \mapsto [\delta_1 a]$.

Now we can obtain a differential $d_2 = \delta_2 j_2^*$, and use this to get the third "page" of the spectral sequence, consisting of terms $A_3^{n,p} = i_2^*(A_2^{n,p})$ and $E_3^{n,p} = H(E_2^{n,p})$ where $H$ denotes homology with respect to $d_2$.

Iteration of this process yields the "spectral sequence" $\{E_k^{n,p}\}$.

## Exact couples

The preceding construction is very messy due to the many terms and many things to check regarding the defined maps on each page. We can simplify this somewhat by the alternate approach of taking direct sums in a way that will leave us with only a few terms to consider.

<u>Def.</u> An exact couple is an exact sequence of the following form:

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "k"] \\ & E \arrow[ul, "j"] & \end{tikzcd}$$

Notice that if we let $A_1 = \bigoplus_{p,q} A_1^{p,q}$ and $E_1 = \bigoplus_{p,q} E_1^{p,q}$, where $A_1^{p,q}$ and $E_1^{p,q}$ are terms from the first page of the Atiyah-Hirzebruch spectral sequence, then $A_1$ and $E_1$ form an exact couple with $i$ induced by $i_1^*$, $j$ induced by $j_1^*$, and $k$ induced by $\delta$.

Given an exact couple, we have a "derived couple":

$$\begin{tikzcd} i(A) \arrow[rr, "i'"] & & i(A) \arrow[dl, "k'"] \\ & H(E) \arrow[ul, "j'"] & \end{tikzcd}$$

where $H(E)$ is the homology at $E$ with respect to the differential $d = kj$, $i' = i|_{i(A)}$, $k' : ia \mapsto [ka]$, and $j' : [e] \to je$.

The following makes a good exercise in diagram chasing:

<u>Proposition</u>. The derived couple of an exact couple is an exact couple.

Now notice that if we let $A_2 = \oplus_{p,q} A_2^{p,q}$ and $E_2 = \oplus_{p,q} E_2^{p,q}$ then

$$\begin{tikzcd} A_2 \arrow[rr, "i_2^*"] & & A_2 \arrow[dl, "\delta_2"] \\ & E_2 \arrow[ul, "j_2^*"] & \end{tikzcd} \quad \text{is the derived couple of} \quad \begin{tikzcd} A_1 \arrow[rr, "i_1^*"] & & A_1 \arrow[dl, "\delta_1"] \\ & E_1 \arrow[ul, "j_1^*"] & \end{tikzcd}$$

## <u>Convergence</u>

If $X$ is a finite-dimensional CW complex, then $\exists R$ s.t. $d_r = 0$ $\forall r > R$. We can then define $E_\infty^{n,p}$ to be the "stable" group $E_{r+1}^{n,p} = E_{r+2}^{n,p} = \dots$

Similarly, $A_1^{n,p}$ and $A_1^{n,-p}$ are stable for large $p$ so that $A_1^{n,\infty} = \tilde{K}^n(X)$ and $A_1^{n,-\infty} = 0$ are well-defined.

<u>Theorem</u>. $E_\infty^{n,p} = \mathrm{Ker}\left(\tilde{K}^n(X) \to \tilde{K}^n(X^{(p+1)})\right) / \mathrm{Ker}\left(\tilde{K}^n(X) \to \tilde{K}^n(X^{(p)})\right)$.

Pf. From the $r$th page of the spectral sequence we get the following exact sequence:

$$\cdots \to E_r^{n+1,p-r+1} \to A_r^{n,p-r+2} \to A_r^{n,p-r+1} \to E_r^{n,p} \to A_r^{n+1,p+1} \to A_r^{n+1,p} \to E_r^{n+1,p+r-1} \to \cdots$$

If we select $r$ to be sufficiently large, then the first three terms in this sequence vanish and we get an isomorphism

$$E_r^{n,p} \cong \mathrm{Ker}\left(A_r^{n+1,p+1} \xrightarrow{i} A_r^{n+1,p}\right) \cong \mathrm{Ker}\left(A_r^{n,\infty} \to A_r^{n+1,p+1}\right) / \mathrm{Ker}\left(A_r^{n+1,\infty} \to A_r^{n+1,p}\right).$$

## <u>Analyzing the Differentials</u>

<u>Proposition</u>. In the Atiyah-Hirzebruch Spectral Sequence, $d_1 : \tilde{K}^n(X^{(p)}/X^{(p-1)}) \to \tilde{K}^{n+1}(X^{(p+1)}/X^{(p)})$ is the cellular coboundary operator, as in the cellular chain complex for cohomology.

To prove this, we'll use the Chern character to identify the K-Theory hom. $d_1$ with the cohomology hom., the coboundary operator.

$$
\begin{tikzcd}
\tilde{K}^n(X^{(p)}/X^{(p-1)}) \arrow[r, "j^*"] \arrow[rr, bend left, "d_1"] \arrow[d, "ch_{\mathbb{Q}}"] \arrow[dd, bend right=60, "ch_{\mathbb{Z}}"'] & \tilde{K}^n(X^{(p)}) \arrow[r, "\delta_1"] \arrow[d, "ch_{\mathbb{Q}}"] & \tilde{K}^{n+1}(X^{(p+1)}/X^{(p)}) \arrow[d, "ch_{\mathbb{Q}}"] \arrow[dd, bend left=60, "ch_{\mathbb{Z}}"] \\
H^*(X^{(p)}/X^{(p-1)}; \mathbb{Q}) \arrow[r] & H^*(X^{(p)}; \mathbb{Q}) \arrow[r] & H^*(X^{(p+1)}/X^{(p)}; \mathbb{Q}) \\
H^*(X^{(p)}/X^{(p-1)}; \mathbb{Z}) \arrow[r] \arrow[u, hook] \arrow[rr, bend right, "d_{\mathbb{Z}}"'] & H^*(X^{(p)}; \mathbb{Z}) \arrow[r] \arrow[u, hook] & H^*(X^{(p+1)}/X^{(p)}; \mathbb{Z}) \arrow[u, hook, "i"']
\end{tikzcd}
$$

To identify these maps, it suffices to show that "the big diagram commutes," i.e. $Ch_{\mathbb{Z}}\, d_1 = d_{\mathbb{Z}}\, Ch_{\mathbb{Z}}$, where $Ch_{\mathbb{Z}}$ is the Chern character whose range is in integral cohomology since $X^{(p)}/X^{(p-1)}$ is homeomorphic to a wedge product of $p$-spheres. Since $i$ is an injection, it then suffices to show that $i\, Ch_{\mathbb{Z}}\, d_1 = i\, d_{\mathbb{Z}}\, Ch_{\mathbb{Z}}$. But $i\, Ch_{\mathbb{Z}}\, d_1 = Ch_{\mathbb{Q}}\, d_1 = d_{\mathbb{Q}}\, Ch_{\mathbb{Q}} = i\, d_{\mathbb{Z}}\, Ch_{\mathbb{Z}}$.

<u>Proposition</u>. The even differential $d_{2r}$ in the Atiyah-Hirzebruch spectral sequence is zero $\forall r$.

Pf. Reindex the first page as follows,

$$\tilde{K}^2(X^{(0)}) \xrightarrow{d_1} \tilde{K}^3(X^{(1)}/X^{(0)}) \xrightarrow{d_1} \tilde{K}^4(X^{(2)}/X^{(1)}) \longrightarrow \cdots$$

$$\tilde{K}^1(X^{(0)}) \xrightarrow{d_1} \tilde{K}^2(X^{(1)}/X^{(0)}) \xrightarrow{d_1} \tilde{K}^3(X^{(2)}/X^{(1)}) \longrightarrow \cdots$$

$$\tilde{K}^0(X^{(0)}) \xrightarrow{d_1} \tilde{K}^1(X^{(1)}/X^{(0)}) \xrightarrow{d_1} \tilde{K}^2(X^{(2)}/X^{(1)}) \longrightarrow \cdots$$

Since, again $X^{(p)}/X^{(p-1)}$ is a wedge product of $p$-spheres, we have that
$$\tilde{K}^n(X^{(p)}/X^{(p-1)}) = \begin{cases} 0 & \text{if } n \equiv p \pmod 2 \\ \bigoplus_{p\text{-cells}} \mathbb{Z} & \text{otherwise} \end{cases}$$

Using this fact, we see that every other row in the reindexed diagram consists of trivial groups. Since an even differential "goes up" an odd number of rows, it follows that even differentials are zero.

Corollary: Atiyah - Hirzebruch spectral sequence existence theorem.
There exists a spectral sequence $\{E_r^{p,q}\}$ with $E_2^{p,q} = H^p(X; \tilde{K}^q(pt))$ and $E_\infty^{p,q} = \operatorname{Ker}(\tilde{K}^p(X) \to \tilde{K}^{p+1}(X^{(q+1)})) / \operatorname{Ker}(\tilde{K}^{p+1}(X) \to \tilde{K}^{p+1}(X^{(q+1)}))$.

Remarks. Many theorems of this form have been proven. As A User's Guide to Spectral Sequences puts it, the prototypical spectral sequence theorem is "there exists a spectral sequence with $E_2$ = 'something computable' and $E_\infty$ = 'something desirable'."
In this case, we show a relationship between K-Theory and cohomology (in fact, the arguments will go through not just for K-Theory but for any cohomology theory).