Lecture 9

Another nice application of the Splitting Principle is the uniqueness of Chern / Stiefel-Whitney classes.

<u>Theorem</u>: The classes w_i, c_i we have defined are the <u>only</u> sequences of real/cplx char. classes satisfying the 3 axioms.

<u>Pf</u>: Let $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a (cplx, say) bdle, and let $X' \xhookrightarrow{f} X$ be a map s.t. f^* is injective on cohomology, and $f^*E = L_1 \oplus - \oplus L_k$ for line bdles $L_1, \ , L_k$. Then if $\beta = 1 + \beta_1 + \cdots + \beta_k$ are char. classes satisfying the axioms, we have

$$f^*(\beta(E)) = \beta(f^*E) = \beta(L_1 \oplus \cdots \oplus L_k) \overset{WSF}{=} \prod_{i=1}^{k} \beta(L_i)$$

$$\underset{\substack{\uparrow \\ \text{axioms} \Rightarrow \\ \beta_2, \beta_3, \text{ vanish on line bdles}}}{=} \prod_{i=1}^{k} (1 + \beta_1(L_i)) \underset{\substack{\uparrow \\ \text{the axioms determine} \\ \text{values on line bdles}}}{=} \prod_{i=1}^{k} (1 + c_1(L_i)) \overset{WSF}{=} c(\oplus L_i)$$

$$= c(f^*E) = f^* c(E).$$

Since f^* is injective, we have $\beta(E) = c(E)$. $\square$

We now establish additivity of the First Stiefel-Whitney Class (Lemma 2, real case). This depends on a new interpretation of $w_1(L) \in H^1(X;\mathbb{Z}/2)$. First, we reinterpret $H^1(X;\mathbb{Z}/2)$.

By the Univ. Coeff. Thm, we have

$$H^1(X;\mathbb{Z}/2) \cong Hom(H_1(X;\mathbb{Z}),\mathbb{Z}/2) \oplus Ext(H_0(X;\mathbb{Z}),\mathbb{Z}/2)$$
$$\cong Hom(H_1(X;\mathbb{Z}),\mathbb{Z}/2)$$

b/c $H_0(X;\mathbb{Z}) = \mathbb{Z}$ is free. Since $H_1(X;\mathbb{Z}) = \pi_1(X)^{ab}$, we find that

$$H^1(X;\mathbb{Z}/2) \cong Hom(\pi_1 X, \mathbb{Z}/2).$$

<u>Claim</u>: The First Stiefel-Whitney class $w_1\left(\begin{smallmatrix} L \\ \downarrow \\ X\end{smallmatrix}\right)$ corresponds to the function $\pi_1 X \xrightarrow{w_1} \mathbb{Z}/2$ defined by

$$w_1([\alpha]) = \begin{cases} 1, & \alpha^*E \text{ is non-trivial} \\ 0, & \text{else} \end{cases}$$

<u>Pf</u>: The function is well-defined by the Bundle Htpy Thm.

First, let's check the formula on the universal line bdle $\begin{smallmatrix} \gamma_1 \\ \downarrow \\ \mathbb{R}P^\infty \end{smallmatrix}$. We know that $\pi_1(\mathbb{R}P^\infty) \cong \pi_1 \mathbb{R}P^2 \cong \mathbb{Z}/2$, so both $H^1(\mathbb{R}P^\infty;\mathbb{Z}/2)$ and $Hom(\pi_1 \mathbb{R}P^\infty, \mathbb{Z}/2)$ have a single non-zero element, which by <u>def'n</u> is $w_1(\gamma_1)$. On the other hand, the pullback of γ_1 along the generator $\alpha: S^1 = \mathbb{R}P^1 \longrightarrow \mathbb{R}P^2$

is precisely the tautological bdle over $\mathbb{RP}^1$, which we have shown is non-trivial. So our new def'n also gives us the unique non-zero map $\pi_1 \mathbb{RP}^\infty \longrightarrow \mathbb{Z}/2$.

To complete the proof, we just note that the new def'n is natural under pullbacks and the diagram

$$\begin{array}{ccc} H^1(X;\mathbb{Z}/2) & \xleftarrow{f^*} & H^1(\mathbb{RP}^\infty;\mathbb{Z}/2) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}(X,\mathbb{Z}/2) & \longleftarrow & \mathrm{Hom}(\pi_1\mathbb{RP}^\infty,\mathbb{Z}/2) \\ \varphi\circ f_* & \longleftarrow\!\shortmid & \varphi \end{array}$$

commutes for any $f: X \to \mathbb{RP}^\infty$.

<u>Proof of Lemma 2</u> (Real Case): Say $L_1 \searrow X \swarrow L_2$ are line bdles. We must show that $w_1(L_1 \otimes L_2) = w_1 L_1 + w_1 L_2$. In terms of maps $\pi_1 X \to \mathbb{Z}/2$, this means we must check that if $N_1 \searrow S^1 \swarrow N_2$ are line bdles, then $N_1 \otimes N_2$ is non-trivial if and only if exactly one of the N_i is non-trivial. The only case to check is that $\gamma_1^1 \otimes \gamma_1^1 \downarrow S^1 = \mathbb{RP}^1$ is trivial. But a choice of metric on γ_1^1 gives an isomorphism $\gamma_1^1 \cong (\gamma_1^1)^*$ (the dual bdle) so $\gamma_1^1 \otimes \gamma_1^1 \cong \gamma_1^1 \otimes (\gamma_1^1)^* \cong S^1 \times \mathbb{R}$. (Note: the last step can also be done by considering the clutching fcns of $\gamma_1^1 \otimes \gamma_1^1$.) $\square$

<u>Rmk</u>: We could take this as <u>our</u> defn of the first Stiefel-Whitney class of line bdles. In fact, it makes sense for <u>any</u> line bdle $\begin{smallmatrix} L \\ \downarrow \\ X \end{smallmatrix}$, even if X is not paracpt (and then L need not be pulled back from $\begin{smallmatrix} \gamma_1 \\ \downarrow \\ \mathbb{R}P^\infty \end{smallmatrix}$).

So in the real case, we don't need to use Milnor's result (MS §5) that bdles over paracpt spaces are pulled back from the universal bdle. (Although we have just shown that this new defn agrees with the old ~~when~~ whenever Milnor's result applies.)

To establish additivity of the first <u>Chern</u> class, we take a more homotopy-theoretical approach. We will show that tensor product gives $\mathbb{C}P^\infty$ a multiplicative structure (up to homotopy), which then induces both tensor product of line bdles and addition of first cohomology classes. (This works equally well for w_1 and $\mathbb{R}P^\infty$.)

To construct a "multiplication" $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$, it suffices to name a line bdle over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ (and then μ will be the classifying map. We will use the bdle $\pi_1^*\gamma_1 \otimes \pi_2^*\gamma_1$, where $\begin{matrix}\gamma_1\\ \downarrow \\ \mathbb{C}P^\infty\end{matrix}$ is the tautological line bdle, and
$$\begin{tikzcd} & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{C}P^\infty & & \mathbb{C}P^\infty \end{tikzcd}$$
are the projections.

<u>Claim</u>: If $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ classifies $\pi_1^*\gamma_1 \otimes \pi_2^*\gamma_1$ (i.e. $\mu^*\gamma_1 \cong \pi_1^*\gamma_1 \otimes \pi_2^*\gamma_1$) then

1) If $i_1 : \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, $x \longmapsto (x, *)$ and $i_2 : \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, $x \longmapsto (*, x)$ are the inclusions, then $\mu i_1, \mu i_2 : \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ are homotopic to the identity (for any $* \in \mathbb{C}P^\infty$);

2) If $\begin{matrix}L\\ \downarrow \\ X\end{matrix}$ is classified by $f : X \to \mathbb{C}P^\infty$ and $\begin{matrix}M\\ \downarrow \\ X\end{matrix}$ is classified by $g : X \to \mathbb{C}P^\infty$, then $\begin{matrix}L\otimes M\\ \downarrow \\ X\end{matrix}$ is classified by $\mu \circ (f, g)$.

__Proof:__ __1)__ To check that $\mu i_1, \mu i_2$ are homotopic to $Id_{\mathbb{CP}^\infty}$, it suffices to check that $(\mu i_1)^* \gamma_1, (\mu i_2)^* \gamma_1$ are isomorphic to $\gamma_1 = Id^*(\gamma_1)$. We have

$$(\mu i_1)^* \gamma_1 \cong i_1^* \mu^* \gamma_1 \cong i_1^* (\pi_1^* \gamma_1 \otimes \pi_2^* \gamma_1)$$
$$\cong (\pi_1 i_1)^* \gamma_1 \otimes \underbrace{(\pi_2 i_1)}_{\text{constant}}{}^* \gamma_1 \cong (Id)^* \gamma_1 \otimes (X \times \mathbb{C})$$
$$\cong \gamma_1,$$

and similarly $(\mu i_2)^* \gamma_1 \cong \gamma_1$.

__Picture:__

$$\begin{tikzcd}
 & \mathbb{CP}^\infty & \\
\mathbb{CP}^\infty \arrow[ur, "\text{constant}"] \arrow[r, "i_1"] \arrow[dr, equal] & \mathbb{CP}^\infty \times \mathbb{CP}^\infty \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[r, "\mu"] & \mathbb{CP}^\infty \\
 & \mathbb{CP}^\infty &
\end{tikzcd}$$

__2)__ If $f^* \gamma_1 = L$, $g^* \gamma_1 = M$, then $(\mu \circ (f,g))^* (\gamma_1) = (f,g)^* (\pi_1^* \gamma_1 \otimes \pi_2^* \gamma_1)$

$$= (\pi_1 \circ (f,g))^* \gamma_1 \otimes (\pi_2 \circ (f,g))^* \gamma_1 = f^* \gamma_1 \otimes g^* \gamma_1 = L \otimes M. \quad \square$$

To compute $c_1(L \otimes M)$, we need to know a bit about the cohomology of $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$. The following result is a weak form of the Kunneth Thm, and is proven in Hatcher (Chap. 3, p. 218-223).

__Kunneth Thm:__ If X, Y are CW cplxs with $H^*(Y;\mathbb{Z})$ torsion-free, then the map

$$H^*(X;\mathbb{Z}) \times H^*(Y;\mathbb{Z}) \longrightarrow H^*(X \times Y;\mathbb{Z})$$
$$\alpha \times \beta \longmapsto \pi_1^* \alpha \cup \pi_2^* \beta$$

induces an isomorphism

$$H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \xrightarrow{\cong} H^*(X \times Y;\mathbb{Z}).$$

Since $\mathbb{C}P^\infty$ is a CW cplx and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[\alpha]$, the theorem applies in our setting.

<u>Theorem</u>: If $L \searrow X \swarrow M$ are line bdles over a paracpt space, then $c_1(L\otimes M) = c_1 L + c_1 M \in H^2(X;\mathbb{Z})$.

<u>Pf</u>: Let $f,g: X \to \mathbb{C}P^\infty$ classify L and M (respectively). Then by the Claim, $\mu\circ(f,g)$ classifies $L\otimes M$, so we just need to show that $(\mu\circ(f,g))^*(\alpha) = f^*\alpha + g^*\alpha$ (where $\alpha \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ is the canonical generator, i.e. $\alpha = c_1(\gamma_1)$).

By the Kunneth Theorem, we can write $\mu^*\alpha = \sum \beta_i \otimes \gamma_i$, where $\deg(\beta_i)+\deg(\gamma_i) = \deg(\mu^*\alpha) = 2$. So since $H^1(\mathbb{C}P^\infty;\mathbb{Z}) = 0$, we in fact can write $\mu^*\alpha = \beta\otimes 1 + 1\otimes\gamma$, with $\beta,\gamma \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$. We claim that $\beta=\gamma=\alpha$: this is where we use Part 1 of the Claim on p. 3.

Letting $i_1 : \mathbb{C}P^\infty \hookrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ be the first inclusion, we have

$$i_1^*(\beta\otimes 1 + 1\otimes\gamma) = i_1^*(\beta\otimes 1) + i_1^*(1\otimes\gamma) \overset{\text{Kunneth isom}}{=\!=} i_1^*(\pi_1^*\beta \cup \pi_2^*(1)) + i_1^*(\pi_1^*1 \cup \pi_2^*\gamma)$$
$$= (i_1^*\pi_1^*\beta)\cup 1 + 1\cup(i_1^*\pi_2^*\gamma) = \beta.$$

But $\mu\circ i_1 \simeq \mathrm{Id}_{\mathbb{C}P^\infty}$, so $i_1^*(\beta\otimes 1 + 1\otimes\beta) = i_1^*\mu^*(\alpha) = \alpha$. So $\beta=\alpha$, and similarly $\gamma = \alpha$. $\square$