Lecture 8

Rmk: Note that since $\deg(c_1(L_E)^k) = 2k$, we have $\deg(c_k(E)) + \deg(c_1(L_E)^{n-k}) = \deg(c_1(L_E)^n)$, i.e. $\deg(c_k E) = 2n - 2(n-k) = 2k$, as expected (and similarly $\deg(w_k(E)) = k$ in the real case).

We now need to check that these classes satisfy the axioms from the Thm on p. 1.

Axiom 1: Grothendieck's Formula only defines the Chern/Stiefel-Whitney classes in the dimensions where they are allowed to be non-zero. We simply extend these def'ns by setting $c_k(E) = 0$ if $k > 2\dim E$, $w_k(E) = 0$ if $k > \dim E$.

Axiom 3: If $\begin{smallmatrix}\gamma\\ \downarrow\\ B\end{smallmatrix}$ is a line bdle, then the projection $\begin{smallmatrix}P(\gamma)\\ \downarrow \pi\\ B\end{smallmatrix}$ is a homeomorphism, and we have a canonical isomorphism $\pi^*\gamma \cong L_\gamma$ of line bdles over $P(\gamma) \cong B$. Grothendieck's formula then defines

$$c_1(L_\gamma) =: c_1(L) \cdot 1_{H^*(P(\gamma);\mathbb{Z})},$$

so our two definitions of c_1 for line bdles agree (and similarly for w_1).

The Whitney Sum Formula (Axiom 2) will take some work, so first we explain naturality. We want the c_i, w_i to be characteristic classes, so we must show that in any diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & X \end{array}$$

we have $c_i(f^*(E)) = f^*(c_i E) \in H^*(B;\mathbb{Z})$ (or $w_i(f^*E) = f^*(w_i(E)) \in H^*(B;\mathbb{Z}/2)$).

This naturality property follows from the fact that projective bdles (and their tautological line bdles) behave well under pullbacks: We have a diagram

$$\begin{tikzcd}
L_{f^*E} \arrow[r, "\cong"] \arrow[d] & (Pf)^*L_E \arrow[r] \arrow[d] & L_E \arrow[d] \\
P(f^*E) \arrow[r, "\cong", "\alpha"'] \arrow[dr] & \tilde{f}^*(PE) \arrow[r, "Pf"] \arrow[d] & PE \arrow[d] \\
& f^*E \arrow[r, "\tilde{f}"] & E
\end{tikzcd}$$

and now the eq'n

$$c_1(L_E)^n = \sum_{i=1}^{n} c_i(E)\, c_1(L_E)^{n-i}$$

pulls back to give

$$\left(\left(Pf\circ\alpha\right)^*(c_1(L_E))\right)^n = \sum_{i=1}^{n} (Pf\circ\alpha)^*(c_i E)\cdot(Pf\circ\alpha)^*(c_1 L_E)^{n-i}.$$

Since c_1 is already natural, we have $(Pf\circ\alpha)^*(c_1 L_E) = c_1(L_{f^*E})$, so the classes $Pf^*(c_i E) \in H^*(Pf^*E) \cong \tilde{f}^*(P_E)$ must be the Chern classes of f^*E (i.e. the unique classes satisfying Grothendieck's formula).

Before proving the Whitney Sum Formula, we need to introduce two more constructions on vector bdles: tensor products and duals. We will follow MS §3, which gives a very general method for extending "continuous" functors on vector spaces to vector bdles.

<u>Def'n</u>: Let <u>Vect</u> denote the category of finite dim'l v. spaces (over $\mathbb{R}$ or $\mathbb{C}$) and <u>isomorphisms</u>. A functor $F: \underline{Vect}^k \longrightarrow \underline{Vect}$ is <u>continuous</u> if each component

function $\mathrm{Isom}(V_1, W_1) \times \cdots \times \mathrm{Isom}(V_k, W_k) \longrightarrow \mathrm{Isom}(F(V_1, \dots, V_k), F(W_1, \dots, W_k))$ is continuous. Note here that if V and W are f.d. v. spaces over $\mathbb{R}$ (or $\mathbb{C}$), then a choice of bases determines a bijection $\mathrm{Hom}(V, W) \xrightarrow{\cong} \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ and the topology on $\mathrm{Hom}(V, W)$ induced by this map is independent of the choice of bases.

Given such a cont. functor F, we want to extend F to a functor

$$\underline{Vect}(X)^k \longrightarrow \underline{Vect}(X),$$

where $\underline{Vect}(X)$ is the category of v. bdles over X.

<u>Def'n</u>: Given $E_1, \dots, E_k \in \underline{Vect}(X)$ and a cont. functor $F: \underline{Vect}^k \to \underline{Vect}$, we define

$$F(E_1, \dots, E_k) = \bigcup_{x \in X} F(p_1^{-1}(x), \dots, p_k^{-1}(x)) \xrightarrow{p} X$$
$$F(\{p_i^{-1}(x)\}) \longmapsto x$$

(here $\begin{matrix} E_i \\ \downarrow p_i \\ X \end{matrix}$ are the projections), with the following topology. Say $U \subseteq X$ is an open set over which each E_i is trivial, and let $\varphi_i : U \times \mathbb{R}^{n_i} \xrightarrow{\cong} E_i|_U$ be trivializations, and let $\varphi_{ix} : \{x\} \times \mathbb{R}^{n_i} \to E_i|_{\{x\}}$ be the restrictions. Then we have $F(\varphi_{1x}, \dots, \varphi_{kx}) : F(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \dots, \mathbb{R}^{n_k}) \xrightarrow{\cong} F(E_1|_x, \dots, E_n|_x)$, which assemble to a fuction

$$\varphi : U \times F(\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_k}) \longrightarrow F(E_1, \dots, E_k)|_U.$$
$$(x, \alpha) \longmapsto F(\varphi_{1x}, \dots, \varphi_{kx})(\alpha)$$

We declare a set $A \subset F(E_1, \ldots, E_k)$ to be open if and only if each preimage $\varphi^{-1}(A)$ is open in $U \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k})$. Since $\varphi^{-1}(\bigcup A_i) = \bigcup \varphi^{-1}(A_i)$ and $\varphi^{-1}(A_1 \cap \cdots \cap A_\ell) = \bigcap_{i=1}^{\ell} \varphi^{-1}(A_i)$, this is a topology.

<u>Lemma</u>: $\begin{array}{c} F(E_1, \ldots, E_k) \\ \downarrow p \\ X \end{array}$ is a (locally trivial) vector bdle.

<u>Pf</u>: To check that p is continuous, it suffices to check that its restriction to each $F(E_1, \ldots, E_k)|_U$ (U a set over which all the E_i are trivial) is continuous. But this is immediate.

Next we will check that each map

$$\varphi : U \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}) \longrightarrow F(E_1, \ldots, E_k)|_U$$

considered above is a homeomorphism. This means we must check that if $W \subset F(E_1, \ldots, E_k)|_U$ is open, then so is $\psi(\varphi^{-1}(W)) \subset U' \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k})$ (where ψ is any other such trivialization). In other words, we must check that $\psi^{-1}\varphi$ is continuous. This follows from continuity of F.

To prove the Whitney Sum Formula, we'll use the 5 following lemmas:

Lemma 1: For any line bdle $\begin{array}{c} L \\ \downarrow p \\ X \end{array}$, the line bdle $\begin{array}{c} L \otimes L^* \\ \downarrow \\ X \end{array}$ is trivial. Here L^* is the dual bundle, constructed in the real case from the functor $V \mapsto \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \overset{\text{def}}{=} V^*$; and in the cplx case from $V \mapsto \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.

Pf: If $v \in p^{-1}(x)$ is any non-zero vector, then set $s(x) = v \otimes v^*$, where $v^* \in \mathrm{Hom}(p^{-1}(x), \mathbb{R})$ sends w to the unique $c \in \mathbb{R}$ s.t. $cw = v$ (i.e. c is the coordinate of w in the basis $\{v\}$). Note that if $c \neq 0$, then $(cv) \otimes (cv)^* = cv \otimes \frac{1}{c}(v^*) = v \otimes v^*$ so s is well-defined and continuous. Since s is never zero, it follows that $L \otimes L^*$ is trivial. The cplx case is the same. $\square$

Lemma 2: The first Chern/SW class is additive on line bdles:

$$c_1(L_1 \otimes L_2) = c_1 L_1 + c_1 L_2 \in H^1(X; \mathbb{Z})$$
$$w_1(L_1 \otimes L_2) = w_1 L_1 + w_1 L_2 \in H^1(X; \mathbb{Z}/2).$$

We postpone the proof.

Proof of the Whitney Sum Formula:

Case 1: Say $E = L_1 \oplus \cdots \oplus L_k$, a sum of line bdles. We will show that $c_1(E) = (1 + c_1 L_1)(1 + c_1 L_2) \cdots (1 + c_1 L_k)$, as expected from iterated application of the WSF. It then follows that $c((L_1 \oplus \cdots \oplus L_k) \oplus (L_1' \oplus \cdots \oplus L_\ell')) = \prod(1 + c_1 L_i) \cdot \prod(1 + c_1 L_i') = c(L_1 \oplus \cdots \oplus L_k) c(L_1' \oplus \cdots \oplus L_\ell')$, proving the WSF for sums of line bdles.

Consider the bdle q^*E, where $\begin{matrix} P(E) \\ \downarrow q \\ X \end{matrix}$ is the projective bdle associated to E. Then there is an injective bdle map $L_E \to q^*E$ (here $L \in P(E)$ and $l \in E$ is a point on L).
$(L,l) \longmapsto (L,l)$

Tensoring with L_E^* gives an injection

$$L_E \otimes L_E^* \hookrightarrow (q^*E) \otimes L_E^* \cong (q^*L_1 \oplus \cdots \oplus q^*L_k) \otimes L_E^*$$
$$\cong (q^*L_1 \otimes L_E^*) \oplus \cdots \oplus (q^*L_k \otimes L_E^*).$$

The section of $L_E \otimes L_E^*$ (Lemma 1) gives a section s of $\bigoplus_{i=1}^k q^*L_i \otimes L_E^*$, and projecting to the factors yields sections s_i of $q^*L_i \otimes L_E^*$. Let $V_i \subseteq P(E)$ be the open set on which s_i is non-zero. Then since $s = \bigoplus_{i=1}^k s_i$ is never zero, we must have $\bigcup_{i=1}^{l} V_i = P(E)$. Now, note that $(q^*L_i \otimes L_E^*)\big|_{V_i}$ is trivial, so $c_1(q^*L_i \otimes L_E^*|_{V_i}) = 0$.

By naturality of c_1, we have $c_1(q^*L_i \otimes L_E^*)\big|_{V_i} = 0$, where $|_{V_i}$ indicates the map on cohomology $H^2PE \to H^2V_i$.
By exactness of the relative cohomology sequences $H^2(PE, V_i) \xrightarrow{j_i} H^2(PE) \to H^2(V_i)$, there exist classes $\gamma_i \in H^2(PE, V_i)$ s.t. $j_i(\gamma_i) = c_1(q^*L_i \otimes L_E^*)$. The relative cup product $\gamma_1 \cup \cdots \cup \gamma_k$ lies in $H^2(PE, \bigcup_{i=1}^k V_i) = H^2(PE, PE) = 0$.
But for any pair $U, V \subseteq Y$, with U, V open, the diagram

$$(\star)\quad \begin{tikzcd} H^*(Y,A)\times H^*(Y,B) \arrow[r, "\cup"] \arrow[d, "j_A\times j_B"] & H^*(Y,A\cup B) \arrow[d, "j_{A\cup B}"] \\ H^*(Y)\times H^*(Y) \arrow[r, "\cup"] & H^*(Y) \end{tikzcd}$$

commutes, where the vertical maps come from the LES of the pairs.

In our case, this says that $j(\gamma_1\cup\cdots\cup\gamma_k)=\prod(j_i\gamma_i)$
$=\prod c_1(q^*L_i\otimes L_E^*)$,

where $j: H^*(PE,PE)\longrightarrow H^*PE$. Since $H^*(PE,PE)=0$, we have $\prod c_1(q^*L_i\otimes L_E^*)=0$.

[<u>Aside</u>: Commutativity of $(\star)$ follows by tracing the def'ns in Hatcher (§3.2, p. 209). The relative cup product is defined via top line in the following diagram:

$$\begin{tikzcd} C^k(Y,A)\times C^\ell(Y,B) \arrow[r, "\cup"] \arrow[d, "j_A\times j_B"] & C^{k+\ell}(Y,A+B) \arrow[r, leftrightarrow, "\text{isom on } H^*"] \arrow[d, "j_{A+B}"] & C^{k+\ell}(Y,A\cup B) \arrow[dl, bend left, "j_{A\cup B}"] \\ C^kY\times C^\ell Y \arrow[r, "\cup"] & C^{k+\ell}(Y) & \end{tikzcd}$$

$C^{k+\ell}(Y,A+B) = \{\varphi: C_{k+\ell}(Y)\to\mathbb{Z} \mid \varphi \text{ vanishes on } C_k(A) \text{ and } C_k B\}$

where $\cup$ in both cases is given by the usual formula:

$\varphi\cup\psi(\sigma)=\varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell}]})$.

It's quick to check this diagram commutes, so $(\star)$ commutes too.]

Continuing, we have $c_1L_E+c_1L_E^* \overset{\text{Lemma 2}}{=\!=} c_1(L_E\otimes L_E^*) \overset{\text{Lemma 1}}{=\!=} 0$, So $c_1(L^*) = -c_1(L)$. Thus Equation (✡) becomes

$$\prod_{i=1}^{k}\Big(\underbrace{c_1(q^*L_i)}_{\substack{\shortparallel \\ q^*(c_1L_i)}} - c_1L_E\Big) = 0,$$

i.e.

$$(c_1L_E)^k = (-1)^{k+1}\left(\sum_{\ell=1}^{k}(-1)^{k-\ell}\Big(\sum_{1\le i_1<\cdots<i_\ell\le k} q^*(c_1L_{i_1})\cup\cdots\cup q^*(c_1L_{i_\ell})\Big)(c_1L_E)^{k-\ell}\right).$$

Hence by our def'n of Chern/Stiefel-Whitney classes, we find that

$$(-1)^{1-\ell}\sum_{1\le i_1<\cdots<i_\ell\le k} q^*(c_1L_{i_1}\cup\cdots\cup c_1L_{i_\ell}) = (-1)^{\ell+1}c_\ell(E)$$

i.e.
$$c_\ell(E) = \sum_{1\le i_1<\cdots<i_\ell\le k} c_1L_{i_1}\cup\cdots\cup c_1L_{i_\ell} \in H^*(X).$$

So $c(E) = 1 + c_1E + \cdots + c_kE = \prod_{i=1}^{k}(1+c_1L_i)$ as claimed.

We now deduce the general case. Given any $\begin{smallmatrix}E\\ \downarrow\\ X\end{smallmatrix}$, the linear injection $L_E \hookrightarrow q^*E$ induces a splitting $q^*E = L_E \oplus L_E^\perp$ (see HW2) of bdles over $P(E)$. We can apply the same principle again, and we find that the pullback of q^*E over $P(q^*E)$ splits as $L_1\oplus L_2\oplus E''$ with L_1, L_2 line bdles. Iterating, we find that there is a map $\tilde{X} \xrightarrow{\pi} X$ such that π^*E is a sum of line bdles, and $\pi^*: H^*X \to H^*\tilde{X}$ is a composite of injections (of the form $q^*: H^*P(\xi) \to H^*Y$ for various bdles $\begin{smallmatrix}\xi\\ \downarrow\\ Y\end{smallmatrix}$).

Now if $E \searrow X \swarrow E'$ are two bdles, let $X_1 \xrightarrow{\pi_1} X$ be such a map for E, and let $X_2 \xrightarrow{\pi_2} X_1$ be such a map for $\pi_1^* E'$.

Then over X_2, we have $(\pi_2 \circ \pi_1)^* E = L_1 \oplus \cdots \oplus L_n$ and $(\pi_2 \circ \pi_1)^* E' = L_1' \oplus \cdots \oplus L_m'$ for some line bdles L_i, L_j'.

Hence by the previous case,

$$c\left((\pi_2 \circ \pi_1)^* E \oplus (\pi_2 \circ \pi_1)^* E'\right) = c\left((\pi_2 \circ \pi_1)^* E\right) \cup c\left((\pi_2 \circ \pi_1)^* E'\right)$$

i.e. $$(\pi_2 \circ \pi_1)^* \left(c(E \oplus E')\right) = (\pi_2 \circ \pi_1)^* \left(c(E) \cup c(E')\right)$$

But $\pi_2 \circ \pi_1^* : H^*(X) \longrightarrow H^*(X_2)$ is injective, so we have

$$c(E \oplus E') = c(E) \cup c(E') \text{ in } H^*(X). \quad \square$$

Rmk: The previous method is known as the Splitting Principle: heuristically, it says that if one wants to derive a formula for all bdls, one just finds a formula that works for sums of line bdles, and then checks that it extends (by the above method). The main pt. is that for every bdle $\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$, there is a map $X' \xrightarrow{f} X$ s.t. $f^* : H^* X' \hookrightarrow H^* X$, and $f^* E$ is a sum of line bdles.