Lecture 7

Axioms for Chern Classes and Stiefel-Whitney Classes

The Stiefel-Whitney classes $w_i(V) \in H^i(B;\mathbb{Z}/2)$ are defined for real vector bdles $\begin{smallmatrix} V \\ \downarrow \\ B \end{smallmatrix}$ (or, equivalently, $Gl_n\mathbb{R}$ or $O(n)$ bdles).

The Chern classes $c_i(V) \in H^{2i}(B;\mathbb{Z})$ are defined for complex vector bdles $\begin{smallmatrix} V \\ \downarrow \\ B \end{smallmatrix}$ (equiv., $Gl_n\mathbb{C}$ or $U(n)$ bdles).

(In both cases, we assume B is paracompact.)

Theorem: There exist unique sequences $c_1, c_2, \cdots$ and $w_1, w_2, \cdots$ of characteristic classes (for $Gl_n\mathbb{R}$ / $Gl_n\mathbb{C}$ bdles, respectively) with $\dim(w_i) = i$, $\dim(c_i) = 2i$, and coeff. gps $\mathbb{Z}/2$ and $\mathbb{Z}$ satisfying the following axioms:

1) $w_i(V) = 0$ for $i > \dim_{\mathbb{R}}(V)$, $w_0(V) = 1 \in H^0(B;\mathbb{Z}/2)$
$c_i(V) = 0$ for $i > \dim_{\mathbb{C}}(V)$, $c_0(V) = 1 \in H^0(B,\mathbb{Z})$.

"Whitney Sum Formula"

2) If V, W are bdles over B, then

$$w_k(V \oplus W) = \sum_{i=0}^{k} w_i(V) \underset{\substack{\uparrow \\ \text{cup product}}}{\cup} w_{k-i}(W) \quad (V, W \text{ real})$$

or

$$c_k(V \oplus W) = \sum_{i+j=k} c_i(V) \cup c_j(W) \quad (V, W \text{ cplx})$$

3) - $w_1\left(\begin{smallmatrix} V_1(\mathbb{R}^2) \\ \downarrow \\ Gr_1(\mathbb{R}^2) \end{smallmatrix}\right) \neq 0$ in $H^1(Gr_1\mathbb{R}^2; \mathbb{Z}/2)$. (Note: $Gr_1\mathbb{R}^2 = \mathbb{R}P^1 = S^1$ and this bdle is the canonical bdle over S^1, and $H^1(S^1;\mathbb{Z}/2) = \mathbb{Z}/2$.)

- $c_1\left(\begin{smallmatrix} V_1(\mathbb{C}^\infty) \\ \downarrow \\ Gr_1(\mathbb{C}^\infty) \end{smallmatrix}\right) \in H^2(Gl_1\mathbb{C}^\infty;\mathbb{Z}) = H^2(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}$ is the class rep'd in cellular cohomology by the cochain sending the unique 2-cell to $1 \in \mathbb{Z}$.

Note that the last axiom rules out the possibility $w_i = c_i = 0$ for all i. Also, since all line bdles over CW cplxs are pulled back from $\mathop{Gr_1(\mathbb{C}^\infty)}^{V_1(\mathbb{R}^\infty)}$, Axiom 3) determines $c_1(L)$ for all line bdles L. To understand the Whitney Sum Formula, we'll need to define the bdle $\begin{array}{c} V\oplus W \\ \downarrow \\ B \end{array}$, whose fiber over $b \in B$ is canonically $V_b \oplus W_b$ (the sum of the fibers).

Note that if we define $w(V) = \sum_{i=0}^{\dim V} w_i(V) \in \bigoplus_{i=0}^{\infty} H^i(B;\mathbb{Z}/2)$ and $c(V) = \sum_{i=0}^{\dim V} c_i(V) \in \bigoplus_{i=0}^{\infty} H^i(B;\mathbb{Z})$, then the Whitney Sum Formula takes the form:

$$w(V\oplus W) = w(V)\cdot w(W) \quad (V, W \text{ real})$$
$$c(V\oplus W) = c(V)\cdot c(W) \quad (V, W \text{ cplx})$$

where the mult'n takes place in the graded ring

$$H^*(B;\mathbb{Z}/2) = \bigoplus_i H^i(B;\mathbb{Z}/2) \text{ or } H^*(B;\mathbb{Z}) = \bigoplus_i H^i(B;\mathbb{Z}).$$

Whitney Sums:

Given two bundles $\begin{array}{ccc} V & & W \\ & {\scriptstyle \pi_V}\searrow \quad \swarrow{\scriptstyle \pi_W} & \\ & B & \end{array}$, we define $\begin{array}{c} V\oplus W \\ \downarrow{\scriptstyle \pi} \\ B \end{array}$ to have total space $V\oplus W = V \times_B W = \{(v,w) \mid \pi_V(v) = \pi_W(w)\}$. The fibers are vector spaces (over $\mathbb{R}$ or $\mathbb{C}$) by component-wise addition and scalar mult'n. If $U \subseteq B$ is an open set over which both V and W are trivial, w/ $\begin{array}{ccc} U\times\mathbb{R}^n & \xrightarrow[\varphi]{\cong} & V|_U \\ & \searrow \quad \swarrow & \\ & B & \end{array}$,

$$\begin{array}{ccc} U\times\mathbb{R}^m \xrightarrow[\psi]{\cong} W|_U & & (u,x,y) \longmapsto (\varphi(u,x), \psi(u,y)) \\ \searrow \quad \swarrow & \text{, then} & U\times\mathbb{R}^n\times\mathbb{R}^m \xrightarrow{(\varphi,\psi)} V\times_B W|_U \\ U & & \searrow \quad U \quad \swarrow \end{array}$$ is a homeomorphism,

b/c $(V \times_B W)|_U = V|_U \times_B W|_U \xrightarrow{(\pi, \varphi_2, \psi_2)} U\times\mathbb{R}^n\times\mathbb{R}^m$ gives the inverse.

[Here $\mathbb{R}^n, \mathbb{R}^m$ can of course be replaced by $\mathbb{C}^n, \mathbb{C}^m$].

There are several other ways to view $V\oplus W$.

- There is a bdle $\begin{array}{c} V\times W \\ \downarrow \\ B_1\times B_2 \end{array}$ associated to any $\begin{array}{c} V \\ \downarrow \\ B_1 \end{array}, \begin{array}{c} W \\ \downarrow \\ B_2 \end{array}$.

  (The topology on $V\times W$ is the product topology, and the bdle is trivial over $U_1\times U_2$ if $V|_{U_1} \cong U_1\times\mathbb{R}^n$, $W|_{U_2} \cong U_2\times\mathbb{R}^m$)

  The Whitney sum is then the pullback
$$\begin{array}{ccc} \Delta^*(V\times W) & \to & V\times W \\ \downarrow & & \downarrow \\ B & \xrightarrow{\Delta} & B\times B. \\ b & \longmapsto & (b,b) \end{array}$$

- If $\{U_i\}_i, \{\varphi_{ji} : U_i\cap U_j \to GL_n\mathbb{R}\}_{i,j}$ and $\{\psi_{ji} : U_i\cap U_j \to GL_m\mathbb{R}\}$ give clutching data for V and W (respectively), then $\{\varphi_{ji}\oplus\psi_{ji} : U_i\cap U_j \to GL_{n+m}(\mathbb{R})\}$ gives clutching data for $V\oplus W$. Here $\varphi_{ji}\oplus\psi_{ji}$ is defined via the block-sum maps $GL_n\mathbb{R}\times GL_m\mathbb{R} \to GL_{n+m}\mathbb{R}$.
$$[A], [B] \longmapsto \begin{bmatrix} [A] & 0 \\ 0 & [B] \end{bmatrix}$$

To see this, we just need to look at the trivializations given above: the transitions for $V\oplus W$ have the form

$$(\varphi_j, \psi_j)^{-1}\circ(\varphi_i, \psi_i) : U_i\cap U_j\times\mathbb{R}^n\times\mathbb{R}^m \to U_i\cap U_j\times\mathbb{R}^n\times\mathbb{R}^m$$
$$u, x, y \longmapsto (u, \varphi_j^{-1}\varphi_i(u,x), \psi_j^{-1}\psi_i(u,y))$$

and the matrix for this transformation at $u\in U_i\cap U_j$ is exactly $\begin{bmatrix} \varphi_{ji}(u) & 0 \\ 0 & \psi_{ji}(u) \end{bmatrix}$.

In MS §3, a general construction is given, which works for other "continuous" functors such as $\otimes$, Hom, etc. It is easy to see that their description of $V \otimes W$ agrees with the first one given above. We'll return to the general construction later.

In order to prove the existence of Stiefel-Whitney and Chern classes, we'll study the cohomology of projective bundles.

Def'n: Let $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$ be a (real or cplx) vector bdle. The projective bundle associated to E is the space $P(E) = (E - E_0) / x \sim cx$ for all $x \in E_0$, $c \in \mathbb{R}$ (or $\mathbb{C}$).
Here E_0, the zero section of E, consists of all the zero vectors.

Lemma: The natural projection $\begin{matrix} P(E) \\ \downarrow \\ B \end{matrix}$ $\begin{matrix} [x] \\ \downarrow \\ \pi(x) \end{matrix}$ is a locally trivial fiber bdle, whose fiber is the projective space $\mathbb{R}P^{n-1}$ or $\mathbb{C}P^{n-1}$ (when V is a $\mathbb{R}^n$- or a $\mathbb{C}^n$-bdle, respectively).

Proof: It suffices to check that $(\mathbb{R}^n - \{0\}) \times U / x \sim cx$ is homeomorphic to $\mathbb{R}P^{n-1} \times U$, but this is immediate. $\square$

Our goal will be to understand the cohomology of projective space bldles (with $\mathbb{Z}$ coeff's in the cplx case, and with $\mathbb{Z}/2$ coeff's in the real case).

These cohomology groups will be described in terms of the Chern / Stiefel-Whitney classes of the tautological line bdle.

Def'n: If $\begin{matrix}E\\\downarrow\\B\end{matrix}$ is a vector bdle, the tautological line bdle $\begin{matrix}\gamma_E\\\downarrow\\P(E)\end{matrix}$ over $P(E)$ is defined by $\gamma_V = \{(L,\vec{v}) \in P(E)\times_B E \mid \vec{v}\in L\} \subseteq P(E)\times E$. (Here we think of points in $P(E)$ as lines through the origin in the fibers of E.)

Note that by def'n, the restriction of γ_E to any fiber of $\begin{matrix}P(E)\\\downarrow\\B\end{matrix}$ is precisely the tautological line bdle on that fiber.

Lemma: $\begin{matrix}\gamma_E\\\downarrow\\P(E)\end{matrix}$ is a locally trivial line bdle over $P(E)$.

Pf: Say $\begin{matrix}P(E)|_U \cong U\times\mathbb{R}P^{n-1}\\\downarrow\\U\end{matrix}$ for some U. If $\begin{matrix}\gamma_1\\\downarrow\\\mathbb{R}P^{n-1}\end{matrix}$ is trivial over $W\subseteq\mathbb{R}P^{n-1}$, then $\begin{matrix}(\gamma_E)|_{U\times W}\\\downarrow\\U\times W\subseteq U\times\mathbb{R}P^{n-1}\end{matrix}$ is trivial as well. □

The Projective Bdle Theorem:

Let $\begin{matrix}P(E)\\\downarrow\pi\\B\end{matrix}$ be the projective bdle associated to a cplx n-plane bdle $\begin{matrix}E\\\downarrow\\B\end{matrix}$.

Then the map $\pi^*: H^n(B;\mathbb{Z}) \to H^n(P(E);\mathbb{Z})$ is injective, and there is an isomorphism of graded $H^*(B;\mathbb{Z})$-modules

$$H^*(B;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[x_1,\dots,x_{n-1}] \xrightarrow[\cong]{\alpha} H^*(P(E);\mathbb{Z}),$$

where $\deg(x_i)=2i$, $\alpha(1\otimes 1)=1$, and $\alpha(1\otimes x_i) = (c_1(L_E))^i$, the i^{th} cup-power of $c_1(L_E)$. In particular, $H^*(P(E);\mathbb{Z})$ is free as an $H^*(B;\mathbb{Z})$-module.

If $\begin{matrix}E\\\downarrow\\B\end{matrix}$ is a real n-plane bdle, we have

$$H^*(B;\mathbb{Z}/2)\otimes_{\mathbb{Z}/2}\mathbb{Z}/2[x_1,\dots,x_{n-1}] \xrightarrow[\alpha]{\cong} H^*(PE;\mathbb{Z}/2) \quad (\text{again } \alpha(1\otimes 1)=1).$$
$$1\otimes x_i \longmapsto w_1(L_E)^i$$

<u>Note</u>: - This theorem <u>does</u> <u>not</u> describe the full ring structure of $H^*(PE)$.

- To make sense of the Chern / Stiefel-Whitney classes of $\begin{matrix} L_E \\ \downarrow \\ PE \end{matrix}$, we need to know that bdles over PE are pulled back from $\begin{matrix} V_1(\mathbb{C}^\infty) \\ \downarrow \\ Gr_1(\mathbb{C}^\infty) \end{matrix}$. If B is paracpt, then so is PE (a simple argument shows that if $\begin{matrix} F \to E \\ \downarrow \\ B \end{matrix}$ is a fiber bdle with F, B paracpt, then E is paracpt as well). So we will need Milnor's theorem (MS Thm 5.6) that all bdles over paracpt base spaces are pulled back from $\begin{matrix} V_n\mathbb{R}^\infty \\ \downarrow \\ Gr_n\mathbb{R}^\infty \end{matrix}$ / $\begin{matrix} V_n\mathbb{C}^\infty \\ \downarrow \\ Gr_n\mathbb{C}^\infty \end{matrix}$, which we'll discuss later. <u>Note that even if B is CW, it's not clear that P(E) is CW.</u>

<u>Grothendieck's Definition of Chern / Stiefel-Whitney Classes</u>:

Given a cplx n-plane bdle $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$, the class $c_1(L_E)^n \in H^*(PE;\mathbb{Z})$ must be expressible in terms of the basis $1, c_1(L_E), \cdots, c_1(L_E)^{n-1}$ for $H^*(PE;\mathbb{Z})$ as an $H^*(B;\mathbb{Z})$-module. In other words, there are unique elements $c_1(E), \cdots, c_n(E) \in H^*(B;\mathbb{Z})$ s.t.

$$(\star) \quad c_1(L_E)^n = (-1)^{n+1} c_n(E) \cdot 1_{H^*(PE;\mathbb{Z})} + (-1)^n c_{n-1}(E) \cdot c_1(L_E) + \cdots + c_1(E) \cdot c_1(L_E)^{n-1}.$$

<u>Rmk</u>: For Stiefel-Whitney classes, the signs have no effect b/c we work with $\mathbb{Z}/2$ coeff's.

<u>Def'n</u>: The class $c_i(E) \in H^*(B;\mathbb{Z})$ appearing in $(\star)$ is the i^{th} Chern class of the bdle E, and the Stiefel-Whitney classes of real n-plane bdles are defined analogously.