So for $* \le k-1$, we have

$$\pi_* V_n(\mathbb{R}^{n+k}) \xrightarrow[(q_n)_*]{\cong} \pi_* V_{n-1}(\mathbb{R}^{n+k}) \xrightarrow[(q_{n-1})_*]{\cong} \pi_* V_{n-2}\mathbb{R}^{n+k} \xrightarrow{\cong} \cdots \xrightarrow{\cong} \pi_* V_1 \mathbb{R}^{n+k}.$$

But $V_1(\mathbb{R}^{n+k}) = S^{n+k-1}$, and $\pi_* S^{n+k-1} = 0$ for $* \le k-1$.

So $\pi_* V_n \mathbb{R}^{n+k} = 0$ for $* \le k-1$, and hence $\begin{matrix} V_n\mathbb{R}^{n+k} \\ \downarrow \\ Gr_n\mathbb{R}^{n+k} \end{matrix}$ is $(k-2)$-universal. When $k=\infty$, it follows that $\pi_* V_n\mathbb{R}^\infty = \pi_* \operatorname{colim}_{k\to\infty} V_n\mathbb{R}^{n+k} \cong \operatorname{colim}_{n\to\infty} \pi_* V_n\mathbb{R}^{n+k} = 0$. $\square$

[Exercise: $\pi_*(\operatorname{colim}(X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots)) \cong \operatorname{colim}_i \pi_* X_i$ if pts in X_i are closed.]

Lecture 6: The Universal Vector Bdle over the Grassmannian:

Defn: The universal bdle $\begin{matrix} \gamma_n(\mathbb{R}^{n+k}) \\ \downarrow \pi \\ Gr_n\mathbb{R}^{n+k} \end{matrix}$ is defined by

$$\gamma_n(\mathbb{R}^{n+k}) = \{ (W, \vec{w}) \in Gr_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \mid \vec{w} \in W \}$$

[We allow $n+k=\infty$, in which case $\mathbb{R}^\infty = \operatorname{colim}_{n\to\infty} \mathbb{R}^n$.]

Claim: $\gamma_n(\mathbb{R}^{n+k})$ is a vector bdle over $Gr_n\mathbb{R}^{n+k}$, with v. space structure inherited from $\mathbb{R}^{n+k}$.

Pf: We'll show that $\gamma_n\mathbb{R}^{n+k}$ is precisely the mixed bdle $V_n(\mathbb{R}^{n+k}) \times_{GL_n\mathbb{R}} \mathbb{R}^n$. We have maps

$$(W, \vec{w}) \longmapsto [\text{basis } \{\vec{w}_i\} \text{ of } W, \text{ coeff. vector of } \vec{w} \text{ in the basis } \{\vec{w}_i\}]$$

$$\gamma_n\mathbb{R}^{n+k} \underset{\alpha^{-1}}{\overset{\alpha}{\rightleftarrows}} V_n \times_{GL_n\mathbb{R}} \mathbb{R}^n$$

$$(W)\vec{x} \longleftarrow [\underbrace{W = (w_1, \cdots w_n)}_{\text{inv'ble } (n+k)\times n \text{ matrix}}, \vec{x}].$$

These maps are inverses of one another, and α^{-1} is clearly continuous. Continuity of α follows from the

Fact that on any open set $\pi^{-1}\mathcal{O}_F = \{(W,\vec{w}) \in \gamma_n\mathbb{R}^{n+k} \mid W \cap \operatorname{Span} F^{\perp} = \{0\}\}$, a chosen basis for Span F determines bases for all $W \in \mathcal{O}_F$ via orthogonal projection. □

__Rmk:__ By an analogous argument, we also have

$$\begin{tikzcd} \gamma_n\mathbb{R}^{n+k} \arrow[d] \arrow[r, "\cong"] & V_n^{\circ}(\mathbb{R}^{n+k}) \times_{O(n)} \mathbb{R}^n \arrow[dl] \\ Gr_n\mathbb{R}^{n+k} & \end{tikzcd}$$

. Hence the Grassmanian inherits a canonical metric.

The __universal bdle__ gets its name from the fact that it classifies vector bdles.

__Theorem:__ Let X be a k-dim'l CW cplx. Then the map

$$[X, Gr_n\mathbb{R}^{n+l}] \longrightarrow \{\mathbb{R}^n\text{-bdles over } X\}/\text{isom}$$
$$[f: X \to Gr_n\mathbb{R}^{n+l}] \longmapsto [f^*\gamma_n\mathbb{R}^{n+k}]$$

is an isomorphism for any $l \geq k$. (We allow $k=\infty$.)

__Rmk:__ MS §5 gives a version of surjectivity for all cpt / paracpt spaces. The above result is neither stronger nor weaker; it covers infinite CW cplxs of finite dim'n.

__Pf:__ We know that $[X, Gr_n\mathbb{R}^{n+l}] \cong \operatorname{Prin}_{GL_n\mathbb{R}}(X)$ for $l \geq \dim X$, and $\operatorname{Prin}_{GL_n\mathbb{R}}(X) \cong \{\mathbb{R}^n\text{- bdles over } X\}/\text{isom}$ by mixing / clutching. Since $\gamma_n\mathbb{R}^{n+k}$ is formed from $V_n\mathbb{R}^{n+l}$ by mixing, the pf is complete. □

Examples: - $Gr_1(\mathbb{R}^n) \cong \mathbb{R}P^{n-1}$, and the universal bdle

$\begin{array}{c} \gamma^1(\mathbb{R}^n) \\ \downarrow \\ Gr_1(\mathbb{R}^n) \end{array}$ is the universal bdle over $\mathbb{R}P^{n-1}$.

- There are canonical homeomorphisms $Gr_n \mathbb{R}^{n+k} \xrightarrow[c]{\cong} Gr_k \mathbb{R}^{n+k}$, given by sending $V \subseteq \mathbb{R}^{n+k}$ to $V^\perp = \{w \mid \langle w, v\rangle = 0 \text{ for all } v \in V\}$.

The bdle $c^*\left(\begin{array}{c}\gamma^k(\mathbb{R}^{n+k}) \\ \downarrow \\ Gr_n \mathbb{R}^{n+k}\end{array}\right)$ is denoted by $\gamma_n(\mathbb{R}^{n+k})^\perp$.

In order to compute $\pi_* V_n^o(\mathbb{R}^{n+k})$, we needed the LES in htpy associated to a fiber bdle. This sequence is best constructed more generally:

Def'n: (Serre) A fibration is a map $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ satisfying the homotopy lifting property: given any (solid) diagram

$$\begin{tikzcd} S^n \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ S^n \times I \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

there exists a lifting $\tilde{H}$ of the htpy H, making the diagram commute.

Lemma (Serre) Every fiber bdle $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ is a fibration.

Rmks: These fibrations are often called Serre fibrations. Any Serre fibration actually satisfies the htpy lifting property with S^n replaced by any CW cplx.

If $\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}$ satisfies the htpy lifting property with S^n replaced by any space, then p is called a Hurewicz fibration. Fiber bdles over reasonable base spaces are Hurewicz fibrations (see [Spanier]).

Fibrations are important because of the following result:

Theorem: Say $\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}$ is a (Serre) fibration, and let $e_0 \in E$ by any basept. Define $p(e_0) =: b_0$, and let $F = p^{-1}(b_0)$ denote the fiber over b_0. Then there are homomorphisms $\partial : \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ forming a LES

$$\cdots \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \xrightarrow{i_*}$$

(where $i : F \hookrightarrow E$ is the inclusion).

Proof of Lemma: Let $\{U_i\}_{i \in I}$ be an open cover of B over which E is trivial. Given a htpy $H : S^n \times I \to B$ and a lifting $F : S^n \times 0 \to E$ of H_0, choose a triangulation of S^n (via subdivision, say) such that for each simplex

$\sigma \in S^n$, $\exists i$ s.t. $H_0(\sigma) \subseteq U_i$. (This is possible b/c the cover $\{H_0^{-1}(U_i)\}_i$ of S^n has positive Lebesgue number, being an open cover of a cpt. set.) We now proceed by induction over the skeleta of S^n. On the zero skeleton, we must extend diagrams

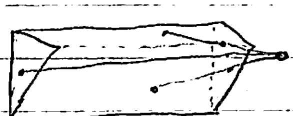
$$\begin{tikzcd} \{0\} \arrow[r, "h_0"] \arrow[d, hook] & E \arrow[d] \\ I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

Choose $0=t_1<\cdots<t_n=1 \in [0,1]$ s.t. each $[t_i,t_{i+1}]$ lies inside $h^{-1}(U_i)$ for some i, and assume we have defined $\tilde{h}$ on $[0,t_k]$. Then on $[t_k,t_{k+1}]$, we know that

$$\begin{tikzcd} E|_{h([t_k,t_{k+1}])} \arrow[d] & {[t_k,t_{k+1}]} \times F \arrow[l, "\varphi_k"', "\cong"] \arrow[dl] \\ h([t_k,t_{k+1}]) & \end{tikzcd}$$

, and we set $\tilde{h}(s) = \varphi_k(s, \pi_2\varphi_k^{-1}(h(t_k)))$ (in other words, we extend $\tilde{h}$ via the retraction $[t_k,t_{k+1}] \to \{t_k\}$).

For the inductive step, we assume $\tilde{h}$ is defined on (k-skeleton of S^n) $\times I$. Given a $(k+1)$-cell $\sigma \subseteq S^n$, we need to extend over $\sigma \times I$. As before we chose $t_i \in I$ s.t. E is trivial over $\sigma \times [t_i, t_{i+1}]$, and we use the retraction $\sigma \times [t_i,t_{i+1}] \to \sigma \times \{t_i\} \cup \partial\sigma \times [t_i,t_{i+1}]$, given by "stereographic projection":



$\square$

Rmk: The proof shows that fiber bdles (and all F.bns) have the HLP wrt all simplicial pairs.

The LES of a fib'n comes directly out of the htpy lifting property.

<u>Pf of Theorem:</u> We begin by checking exactness of

$\pi_n F \xrightarrow{i_*} \pi_n E \xrightarrow{p_*} \pi_n B$. Since pi is constant,

$p_* i_* = 0$. On the other hand, if $\alpha : S^n \to E$ and

$p_*(\alpha) = 0$, then we have a htpy $S^n \times I \xrightarrow{H} B$ with

$H(* \times I \cup S^n \times \{1\}) = b_0$ ($* \in S^n$ is the basepoint). This

gives a diagram

constant map at e_0

$$\begin{tikzcd} S^n \times 0 \cup * \times I \arrow[r, "\alpha \cup c_{e_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\ S^n \times I \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

so a lift $\tilde{H}$ exists.

Since $p \circ \tilde{H}_1 = H_1 = c_{b_0}$ (the constant map at b_0) we see

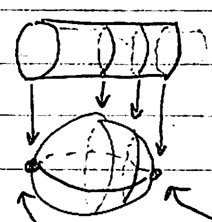
that $\tilde{H}_1 : S^n \to E$ actually lands in F, and $[\tilde{H}_1] \in \pi_n F$

satisfies $i_*[\tilde{H}_1] = [\alpha]$ (b/c $\tilde{H}$ is a htpy from α to $\tilde{H}_1$).

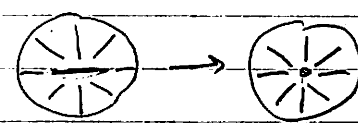
Next we must construct the boundary map $\partial : \pi_n B \to \pi_{n-1} E$.

We begin with two observations regarding spheres. First,

we have a homeomorphism $S^{n-1} \times I / _{S^{n-1} \times 0,\ S^{n-1} \times 1} \cong S^n$,

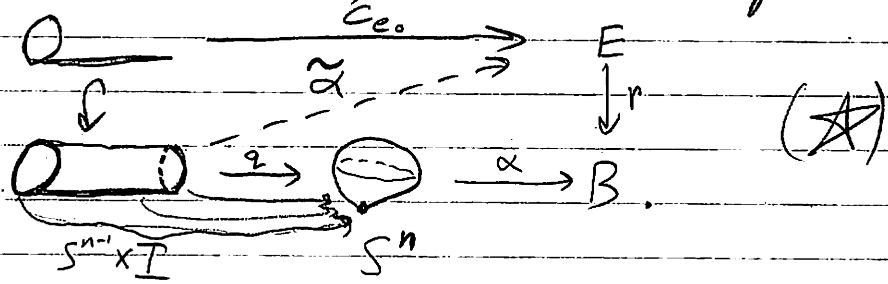
given by vertical projection: . Next,

if $I \subset S^n$ denotes an arc between the pts and, then $S^n \cong S^n / I$:

the homeomorphism descends from the map .

which collapses an interval in D^n while fixing the bdry.

Now, given $\alpha : S^n \to B$, we consider the diagram



We know a lift $\tilde{\alpha}$ exists and $\tilde{\alpha}_1 : S^{n-1} \to E$ actually lands in $F = p^{-1}(b_0)$ b/c $q(S^{n-1}\times\{1\}) = * \in S^n$.

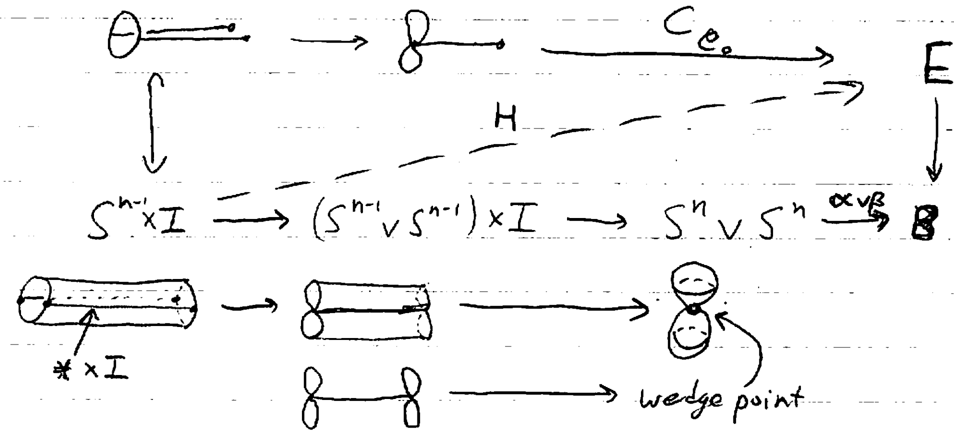
We define $\partial[\alpha] = [\tilde{\alpha}_1]$. We must check that this is a well-defined homomorphism.

<u>Well-defined</u>: Say $H : S^n \times I \to B$ is a htpy from α to β (with $H(* \times I) = b_0$), and say $\tilde{\alpha}$ and $\tilde{\beta}$ are lifts as in diagram $(\star)$. We must show that $\tilde{\alpha} \simeq \tilde{\beta}$. Consider the diagram

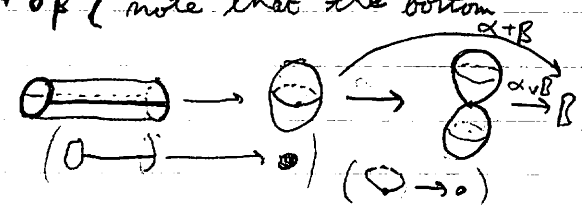
$$\begin{tikzcd}
(S^{n-1}\times I)\times\{0,1\} \cup (* \times I)\times I \arrow[r, "\tilde{\alpha}\cup\tilde{\beta}\cup c_{e_0}"] \arrow[d, hook] & & E \arrow[d] \\
(S^{n-1}\times I)\times I \arrow[r, "q\times 1_I"] \arrow[urr, dashed, "\tilde{H}"] & S^n \times I \arrow[r, "H"] & B.
\end{tikzcd}$$

The lift $\tilde{H}$ exists, b/c we can apply the HLP to the pair $(S^{n-1}\times\{0\}\times I,\ S^{n-1}\times\{0\}\times\{0,1\} \cup \{*\}\times\{0\}\times I)$, treating the <u>first</u> I coordinate as the htpy coordinate. We now see that $\tilde{H}_1 : S^{n-1}\times\{1\}\times I \to E$ lies in F, and is a htpy from $\tilde{\alpha}_1$ to $\tilde{\beta}_1$ (preserving the basept).

Homomorphism: Given $\alpha, \beta : S^n \to B$, we must find a htpy b/w $\partial\alpha + \partial\beta$ and $\partial(\alpha+\beta)$. Note that since ∂ is well-defined, we can form these elements using any lifts we like. Consider the diagram



The map H factors through $(S^{n-1} \vee S^{n-1}) \times I$, and at time 1 it is both $\partial(\alpha+\beta)$ and $\partial\alpha + \partial\beta$ (note that the bottom map in the diagram factors through



Exactness of $\pi_n B \xrightarrow{\partial} \pi_{n-1} F \xrightarrow{i_*} \pi_{n-1} E$ and $\pi_n E \xrightarrow{p_*} \pi_n B \xrightarrow{\partial} \pi_{n-1} F$ are exercises of a similar nature. $\square$

This completes the proof that the Stiefel mflds are universal bdles over the Grassmannians.

We now turn to the construction of important characteristic classes.