

Lecture 5

Characteristic Classes:

Defn: A characteristic class for principal G -bundles is a natural transformation

$$\text{Prin}_G(X) \xrightarrow{c} H^n(X; A),$$

defined for all CW complexes X (n and A are fixed; c are the dim'n and coeff. gp for the class).

Naturality just means that given a map $f: X \rightarrow Y$

and a bdl $\begin{array}{c} P \\ \downarrow \\ Y \end{array}$, we have $c(f^*Y) = f^*(c(P)) \in H^n(X; A)$.

Lemma (Yoneda):

Say $\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$ is a universal principal G -bundle w/ B a CW cplx.

Then there is a bijection

$$\left\{ \begin{array}{l} \text{Char. classes for } G\text{-bundles} \\ \text{w/ coeff. gp. } A \text{ and dim'n } n \end{array} \right\} \xrightarrow{\cong} H^n(BG; A).$$

PF: The bijection is given by $c \mapsto c(EG) \in H^n(BG; A)$.

It remains only to note that for any $\alpha \in H^n(BG; A)$,

the assignment $P \mapsto f^*(\alpha)$ (where f is a classifying map for P) defines a characteristic class; this uses

the bijection $\text{Prin}_G(X) \cong [X, BG]$. □

Our next goal will be to construct universal

principal G -bdles $\begin{matrix} EG \\ \downarrow \\ BG \end{matrix}$. Some of this material is in MS 85.

General Constructions:

- For any group G , Milnor constructs a bdl $\begin{matrix} EG \\ \downarrow \\ BG \end{matrix}$ where $EG = \ast_{i=1}^{\infty} G^{i+1}$ ("infinite join"). Topologized properly, $EG \simeq \ast$ and $\begin{matrix} EG \\ \downarrow \\ BG = EG/G \end{matrix}$ is a principal G -bdle. This construction is functorial in G . Moreover, it classifies bdl's over all paracomp Hausd. spaces, not just CW cplx's.
- Unclear when this BG is a CW cplx.
- Related constructions: Segal, Milgram.

We'll consider specific constructions for the gps of interest to us: $GL_n \mathbb{R}$, $O(n)$, $GL_n \mathbb{C}$, $U(n)$. The cplx case is nearly identical to the real case, so we'll work over \mathbb{R} (and note the differences).

Def'n: The Stiefel manifold $V_n(\mathbb{R}^{n+k}) \subseteq \underbrace{\mathbb{R}^{n+k} \times \dots \times \mathbb{R}^{n+k}}_n$

is the subspace consisting of all linearly indep. n -tuples $(\vec{x}_1, \dots, \vec{x}_n)$.

(Note that $V_n(\mathbb{R}^{n+k})$ is open in $(\mathbb{R}^{n+k})^n$) Such n -tuples are called n -frames.

The Grassmannian $G_n(\mathbb{R}^{n+k})$ is the quotient space

$V_n(\mathbb{R}^{n+k}) / (\vec{x}_1, \dots, \vec{x}_n) \sim (\vec{y}_1, \dots, \vec{y}_n)$ if $\text{Span}\{\vec{x}_i\} = \text{Span}\{\vec{y}_i\}$. Hence $G_n(\mathbb{R}^{n+k})$ consists

of all n -dim'd subspaces of \mathbb{R}^{n+k} . Note that $GL_n \mathbb{R}$

acts on $V_n(\mathbb{R}^{n+k})$ by $(\vec{x}_1, \dots, \vec{x}_n) \cdot A = (\vec{x}_1 \cdot A, \dots, \vec{x}_n \cdot A)$ and $V_n(\mathbb{R}^{n+k}) / GL_n \mathbb{R} \cong G_n(\mathbb{R}^{n+k})$.

There is a second version of the Stiefel mfd, involving

orthonormal (o.n.) frames:

Def'n: $V_n^o(\mathbb{R}^{n+k}) = \{(\bar{x}_1, \dots, \bar{x}_n) \in (\mathbb{R}^{n+k})^n \mid \langle x_i, x_j \rangle = \delta_{ij}\} \subseteq (\mathbb{R}^{n+k})^n$

Note that $O(n)$ acts on $V_n^o(\mathbb{R}^{n+k})$ and (again) $V_n^o(\mathbb{R}^{n+k})/O(n) \cong Gr_n \mathbb{R}^{n+k}$.
 (The inclusion $V_n^o \mathbb{R}^{n+k} \hookrightarrow V_n \mathbb{R}^{n+k}$ induces a cont. bij. $V_n^o \mathbb{R}^{n+k}/O(n) \xrightarrow{\cong} Gr_n \mathbb{R}^{n+k}$;
 it's a homeomorphism b/c, the Gram-Schmitt process gives a cont. map
 $V_n \mathbb{R}^{n+k} \xrightarrow{GS} V_n^o \mathbb{R}^{n+k}$ and α^{-1} is given by $V_n \mathbb{R}^{n+k} \xrightarrow{GS} V_n^o \mathbb{R}^{n+k}$
 $\downarrow \quad \downarrow$
 $Gr_n \mathbb{R}^{n+k} \xrightarrow{\alpha^{-1}} V_n^o(\mathbb{R}^{n+k})/O(n)$

Def'n: The infinite Stiefel mfd's $V_n(\mathbb{R}^\infty), V_n^o(\mathbb{R}^\infty)$

are defined by $V_n(\mathbb{R}^\infty) = \text{colim}_{k \rightarrow \infty} V_n(\mathbb{R}^{n+k}), V_n^o(\mathbb{R}^\infty) = \text{colim}_{k \rightarrow \infty} V_n^o(\mathbb{R}^{n+k})$,

and $Gr_n(\mathbb{R}^\infty) := V_n(\mathbb{R}^\infty)/GL_n \mathbb{R} \cong V_n^o(\mathbb{R}^\infty)/O(n) \cong \text{colim}_{k \rightarrow \infty} Gr_n(\mathbb{R}^{n+k})$.

Here $\text{colim}(X_1 \hookrightarrow X_2 \hookrightarrow \dots)$ is topologized by declaring

$U \subset \text{colim } X_i$ to be open $\iff U \cap X_i$ is open for all i .

Theorem 1: The bundles $V_n(\mathbb{R}^\infty) \downarrow Gr_n(\mathbb{R}^\infty)$ and $V_n^o(\mathbb{R}^\infty) \downarrow Gr_n(\mathbb{R}^\infty)$ are universal principal $GL_n \mathbb{R}/O(n)$ bdl's. Moreover, $V_n(\mathbb{R}^{n+k})$ and $V_n^o(\mathbb{R}^{n+k})$ are $(k-2)$ -universal [i.e. $\pi_* V_n \mathbb{R}^{n+k} = \pi_* V_n^o \mathbb{R}^{n+k} = 0$ for $* \leq k-1$]

(MS6) Theorem 2: $Gr_n(\mathbb{R}^{n+k})$ and $Gr_n(\mathbb{R}^\infty)$ are CW cplx's.

[Note: In the cplx case, Thm 1 would state that these bdl's are $(2k-1)$ -universal, i.e. $\pi_* V_n \mathbb{C}^{n+k} \cong \pi_* V_n^o \mathbb{C}^{n+k} = 0$ for $* \leq 2k-1$]

To prove Theorem 1, we'll use:

Lemma (MW 1): If G acts freely on P , then $\frac{P}{G}$ is a principal

G -bundle \Leftrightarrow 1) π admits local sections
 2) $\frac{P \times P}{G} \rightarrow G$ is continuous.
 $(p, p') \mapsto \left(\begin{array}{l} \text{the unique } g \in G \\ \text{s.t. } p \cdot g = p' \end{array} \right)$

To find local sections, we'll use the following open cover of $Gr_n(\mathbb{R}^{n+k})$:

Def'n: Given any frame $F \in V_n(\mathbb{R}^{n+k})$, we set
 $\mathcal{O}_F = \{ W \in Gr_n(\mathbb{R}^{n+k}) \mid W \cap \text{Span}(F)^\perp = \{0\} \}$. (We allow $n+k = \infty$)

Claim: Each \mathcal{O}_F is open, and $\{ \mathcal{O}_F \mid F \in V_n(\mathbb{R}^{n+k}) \}$ covers $Gr_n(\mathbb{R}^{n+k})$ for any $k \leq \infty$

Pf: For any $W \in Gr_n(\mathbb{R}^{n+k})$, and any o.n. basis $\{w_1, \dots, w_n\}$ for W ,

we have $W \cap \text{Span}(w_1, \dots, w_n)^\perp = W \cap W^\perp = \{0\}$ so $W \in \mathcal{O}_{(w_1, \dots, w_n)}$.

To see that \mathcal{O}_F is always open, note that its inverse image in $V_n(\mathbb{R}^{n+k})$ is $\{ H \in V_n(\mathbb{R}^{n+k}) \mid \text{Span } H \cap (\text{Span } F)^\perp = \{0\} \}$.

Now, if $H = \{ \vec{h}_1, \dots, \vec{h}_n \}$ and $(\vec{f}_1, \dots, \vec{f}_k)$ is a basis for $(\text{Span } F)^\perp$,
 then $\text{Span } H \cap (\text{Span } F)^\perp = \{0\} \Leftrightarrow \det(\vec{h}_1, \dots, \vec{h}_n, \vec{f}_1, \dots, \vec{f}_k) \neq 0$.

(When $k = \infty$, this shows that $\mathcal{O}_F \cap Gr_n(\mathbb{R}^{n+k})$ is open in $Gr_n(\mathbb{R}^{n+k})$ for each $k < \infty$, which implies \mathcal{O}_F is open in $Gr_n(\mathbb{R}^\infty)$.)

Claim: The projections $V_n(\mathbb{R}^{n+k}) \xrightarrow{\pi} \text{Gr}_n(\mathbb{R}^{n+k})$ and $V_n^0(\mathbb{R}^{n+k}) \xrightarrow{\pi^0} \text{Gr}_n(\mathbb{R}^{n+k})$ both admit

sections over each open set Θ_F (w/ $F \in V_n^0(\mathbb{R}^{n+k})$) in the case of $\bar{\pi}$.

[Rmk: A thm of Andrew Gleason says that actions of cpt Lie grps always admitslices.] (the space must be regular)

PF: Given $F = (\bar{x}_1, \dots, \bar{x}_n)$, set $W = \text{Span}(\bar{x}_i)$.

Then if $V \in \Theta_F$, elementary linear algebra shows

that the orthogonal projection $\text{proj}_{V,W}: W \rightarrow V$ is a

linear isomorphism. We define $S_F(V) = \left(\text{proj}_{V,W}(\bar{x}_i) \right)_{i=1}^n \in V_n(\mathbb{R}^{n+k})$,

and we define $\bar{S}_F(V) = \text{GS} \left(\left(\text{proj}_{V,W} \bar{x}_i \right)_{i=1}^n \right) \in V_n^0(\mathbb{R}^{n+k})$,

where GS denotes the Gram-Schmidt orthogonalization process.

Then $\pi S_F = \pi^0 \bar{S}_F = \text{id}_{\text{Gr}_n(\mathbb{R}^{n+k})}$. To check continuity we

just need to check that the compariter $S_F \circ \pi^0, \bar{S}_F \circ \pi^0$ are continuous.

But $S_F \circ \pi^0(\bar{u}_1, \dots, \bar{u}_n) = \left(\text{proj}(\bar{x}_i) \right)_{i=1}^n = \left(\sum_{j=1}^n \langle u_j, \bar{x}_i \rangle u_j \right)_{i=1}^n$

for any o.n. n-frame $\{u_j\}_{j=1}^n$, which is clearly cont. in the \bar{u}_j ,

and $\bar{S}_F \circ \pi^0 = \text{GS} \circ S_F \circ \pi^0$. \square

To complete the proof that the Stiefel mflds are principal bldrs over $\text{Gr}_n(\mathbb{R}^{n+k})$, we need to check condition (2) in the lemma.

Claim: The map: $V_n \mathbb{R}^{n+k} \times_{Gr_n \mathbb{R}^{n+k}} V_n \mathbb{R}^{n+k} \rightarrow GL_n \mathbb{R}$
 $[V], [W] \mapsto [A] \text{ s.t. } [V][A] = [W]$
 (and the corresponding map for o.n. frames) are continuous.

Proof: When the matrix $[A]$ exists, it is given by

$$[A] = [V]^* [W] \text{ for any right inverse } [V]^* \text{ to } [V].$$

(Note that mult'n by the $(n+k) \times n$ matrix $[V]$ is a linear injection $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$.)

This left inverse $[V]^*$ can be chosen continuously, as was

shown in the proof of local triviality for tangent bdl's. \square

Corollary: $V_n(\mathbb{R}^{n+k}) \xrightarrow{\pi} Gr_n \mathbb{R}^{n+k} \xleftarrow{\pi^0} V_n^0(\mathbb{R}^{n+k})$ are principal bdl's.

To prove universality, we need to compute

the hpy gpr $\pi_* V_n(\mathbb{R}^{n+k}), \pi_* V_n^0 \mathbb{R}^{n+k}$.

Exercise: The Gram-Schmidt process defines a deformation retraction
 $V_n(\mathbb{R}^{n+k}) \xrightarrow{\sim} V_n^0(\mathbb{R}^{n+k})$, or $\pi_* V_n \mathbb{R}^{n+k} \cong \pi_* V_n^0 \mathbb{R}^{n+k}$.

We'll study $\pi_* V_n^0 \mathbb{R}^{n+k}$ via the following decomposition.

Prop'n: The map $V_n^0 \mathbb{R}^{n+k} \xrightarrow{\cong} V_n^0 \mathbb{R}^{n+k}$ is a (locally
 $(\tilde{u}_1, \dots, \tilde{u}_{n+k}) \rightarrow (\tilde{u}_1, \dots, \tilde{u}_n)$
 trivial) fiber bdl with fiber $S^{k-1} = V_1^0(\mathbb{R}^k)$.

[Note: in the cplx case, the fiber is $V_1^0(\mathbb{C}^k) = S^{2k-1}$.]

Pf: We'll show that this map is trivial over each

open set $\pi^{-1} \mathcal{O}_F \subseteq V_n^0 \mathbb{R}^{n+k}$, where $V_n^0 \mathbb{R}^{n+k} \xrightarrow{\pi} Gr_n \mathbb{R}^{n+k}$.

We define $\pi^{-1}\mathcal{O}_F \times V_1^0(\text{Span } F^\perp) \xrightarrow{\alpha} q^{-1}\mathcal{O}_F$ by
 $\alpha((\vec{u}_1, \dots, \vec{u}_n), \vec{u}_{n+1}) = GS(\vec{u}_1, \dots, \vec{u}_n, \vec{u}_{n+1})$.

Note that since $\vec{u}_1, \dots, \vec{u}_n$ are already o.n., GS does not affect them, and $\{\vec{u}_1, \dots, \vec{u}_{n+1}\}$ is linearly independent because $\vec{u}_{n+1} \in (\text{Span } F)^\perp$ and $\text{Span}(\vec{u}_1, \dots, \vec{u}_n) \in \mathcal{O}_F$ (or $\text{Span}(\vec{u}_1, \dots, \vec{u}_n) \cap \text{Span } F^\perp = \{0\}$).

It remains to check that $\alpha^{-1}: q^{-1}\mathcal{O}_F \rightarrow \pi^{-1}\mathcal{O}_F \times V_1^0(\text{Span } F^\perp)$ is continuous. But $\alpha^{-1}(\vec{u}_1, \dots, \vec{u}_n, \vec{u}_{n+1}) = (\vec{u}_1, \dots, \vec{u}_n, \frac{\text{proj}_{\text{Span } F^\perp}(\vec{u}_{n+1})}{\|\text{proj}_{\text{Span } F^\perp}(\vec{u}_{n+1})\|})$.

□

To compute h_{top} , we need:

Theorem: For any fiber bundle $F \xrightarrow{i} E \xrightarrow{p} B$ there exists a LES $\dots \rightarrow \pi_* F \xrightarrow{i_*} \pi_* E \xrightarrow{p_*} \pi_* B \xrightarrow{\partial} \pi_{*-1} F \xrightarrow{i_*} \dots \rightarrow \pi_0 B$.

Using this fact, we can complete the proof of universality.

Proof of Thm 1: By the Cellular Approx. Thm, all maps

$S^m \rightarrow S^k$ are null homotopic for $m < k$, i.e. $\pi_m S^k = 0$ for $m < k$.

The LES for the bundle $S^k \rightarrow V_n^0(\mathbb{R}^{n+k}) \xrightarrow{q} V_{n+1}^0(\mathbb{R}^{n+k})$ now has the form

$0 \rightarrow \pi_* V_n^0(\mathbb{R}^{n+k}) \xrightarrow{q_*} \pi_* V_{n+1}^0(\mathbb{R}^{n+k}) \rightarrow 0$ for $* \leq k-1$.

So for $* \leq k-1$, we have

$$\pi_* V_n(\mathbb{R}^{n+k}) \xrightarrow{(q_n)_*} \pi_* V_{n-1}(\mathbb{R}^{n+k}) \xrightarrow{(q_{n-1})_*} \pi_* V_{n-2}(\mathbb{R}^{n+k}) \xrightarrow{\dots} \pi_* V_1(\mathbb{R}^{n+k})$$

But $V_1(\mathbb{R}^{n+k}) = S^{n+k-1}$, and $\pi_* S^{n+k-1} = 0$ for $* \leq k-1$.

So $\pi_* V_n \mathbb{R}^{n+k} = 0$ for $* \leq k-1$, and hence $V_n \mathbb{R}^{n+k}$ is $(k-1)$ -universal. When $k = \infty$, it follows that $\pi_* V_n \mathbb{R}^\infty = \pi_* \text{colim}_{k \rightarrow \infty} V_n \mathbb{R}^{n+k} \cong \text{colim}_{n \rightarrow \infty} \pi_* V_n \mathbb{R}^{n+k} = 0$. \square

[Exercise: $\pi_* (\text{colim}(X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \dots)) \cong \text{colim} \pi_* X_i$ if pts in X_i are closed.]

Lecture 6: The Universal Vector Bundle over the Grassmannian:

Defn: The universal bundle $\gamma_n(\mathbb{R}^{n+k})$ is defined by

$$\begin{array}{c} \gamma_n(\mathbb{R}^{n+k}) \\ \downarrow \pi \\ \text{Gr}_n \mathbb{R}^{n+k} \end{array}$$

$$\gamma_n(\mathbb{R}^{n+k}) = \{ (W, \vec{w}) \in \text{Gr}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \mid \vec{w} \in W \}$$

[We allow $n+k = \infty$, in which case $\mathbb{R}^\infty = \text{colim}_{n \rightarrow \infty} \mathbb{R}^n$.]

Claim: $\gamma_n(\mathbb{R}^{n+k})$ is a vector bundle over $\text{Gr}_n \mathbb{R}^{n+k}$, with v. space structure inherited from \mathbb{R}^{n+k} .

PF: We'll show that $\gamma_n \mathbb{R}^{n+k}$ is precisely the mixed

bundle $V_n(\mathbb{R}^{n+k}) \times_{\text{GL}_n \mathbb{R}} \mathbb{R}^n$. We have maps

$$\begin{array}{ccc} (W, \vec{w}) & \xrightarrow{\alpha} & [\text{basis } \{\vec{w}_i\} \text{ of } W, \text{ coeff. vector of } \vec{w} \text{ in the basis } \{\vec{w}_i\}] \\ \gamma_n \mathbb{R}^{n+k} & \xrightarrow{\alpha} & V_n \times_{\text{GL}_n \mathbb{R}} \mathbb{R}^n \\ (W) \vec{x} & \xleftarrow{\alpha^{-1}} & [(W) = (w_1, \dots, w_n), \vec{x}] \\ & & \text{invertible } (n+k) \times n \text{ matrix} \end{array}$$

These maps are inverses of one another, and α^{-1} is clearly continuous. Continuity of α follows from the