In the other direction, given $\begin{array}{c} P\times_G E \\ s\big(\big\downarrow\pi \\ P/G=X \end{array}$, we define $P \xrightarrow{\varphi} E$ as follows:

We can always write $s([p]) = [p, e]$ for some (unique) $e\in E$ and we set $\varphi(p) = e$. Continuity of φ can be checked locally, using the fact that $(U\times G)\times_G E \xrightarrow{\cong} U\times E$. $\square$

$$[u,g,e] \longmapsto (u, eg^{-1})$$
$$[u,1,e] \longleftarrow (u,e)$$

Lecture 4: 1

Prop'n: Let X be a CW cplx, and let $\begin{array}{c}E\\ \downarrow \\ X\end{array}$ be a fiber bundle with fiber F. If $\pi_* F = 0$ for $* \leq \dim X$ then E admits a section (if $\dim X = \infty$, we just require $\pi_* F = 0$ for $* \geq 0$).

In particular, the mixed bdles $\begin{array}{c}P\times_G EG\\ \downarrow \\ X\end{array}$ always admit sections, so we obtain the required G-map $P \to EG$.

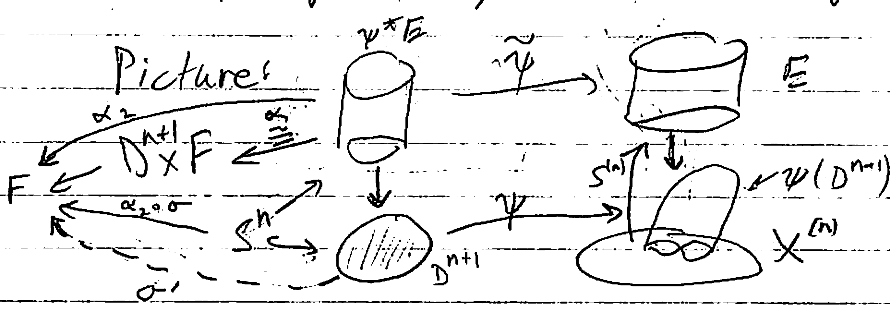
In the proof, we will need to use the fact that every principal G-bdle over a contractible space is trivial. This follows from the Bundle Htpy Theorem, which we'll prove next time.

<u>Proof of Prop'n:</u> By induction on skeleta. Certainly we can define a section $s \curvearrowright \begin{smallmatrix} E \\ \downarrow \\ X^{(0)} \end{smallmatrix}$. Now assume we have a section $s^{(n)}$ defined on $X^{(n)}$; we must extend it over each $(n+1)$-cell. Let $\psi: D^{n+1} \to X$ be the characteristic map for an $(n+1)$-cell in X, so $\psi|_{S^n}: S^n \to X^{(n)}$. Then since $D^{n+1} \simeq *$, the pullback $\psi^* E$ is a trivial bdle over D^{n+1}. The composite $S^n \xrightarrow{\psi} X^{(n)} \xrightarrow{s^{(n)}} E$ gives us a section of $\sigma \curvearrowright \begin{smallmatrix} \psi^* E \\ \downarrow \\ S^n \end{smallmatrix}$: $\sigma(x) = (x, s^{(n)}\psi(x))$. Since $\psi^* E \underset{\alpha}{\cong} D^{n+1} \times F$, we get a corresponding map $\alpha_2 \circ \sigma: S^n \to F$ which extends to $\sigma': D^{n+1} \to F$ (b/c $\pi_{n+1} F = 0$). Letting $\tilde{\sigma}: D^n \to E$ be the map $\tilde{\sigma}(x) = \tilde{\psi}\alpha^{-1}(x, \sigma'(x))$ (where $\begin{smallmatrix} \psi^* E & \xrightarrow{\tilde{\psi}} & E \\ \downarrow & & \downarrow \\ D^{n+1} & \xrightarrow{\psi} & X \end{smallmatrix}$) the desired ext'n of $s^{(n)}$ is:

$$D^{n+1} \cup_{\psi} X^{(n)} \xrightarrow{\tilde{\sigma} \cup s^{(n)}} E$$

(tracing the defns shows that $\tilde{\sigma}(x) = s^{(n)}\psi(x)$ for $x \in S^n$, so this is well-defined and continuous).

Picture:



$\square$

A key tool for studying bundles is the following result:

<u>The Bundle Homotopy Theorem</u>: Let B be a paracompact Hausdorff space, and let $\begin{smallmatrix} E \\ \downarrow p \\ B\times I \end{smallmatrix}$ be a principal G-bdle. Then there is a bundle isomorphism

$$\begin{tikzcd} E \arrow[rr, "\cong"] \arrow[dr] & & E_0 \times I \arrow[dl] \\ & B\times I & \end{tikzcd}$$

, where $E_0 = p^{-1}(B\times\{0\})$.

[The corresponding result for vector bdles/Euclidean bdles follows by clutching/mixing.]

<u>Rmk</u>: This isomorphism restricts to give isomorphisms $E_t \cong E_0$ for all $t\in I$, where $E_t = p^{-1}(B\times\{t\})$. In particular, $E_0 \cong E_1$, so we can think of the BHT as saying that "homotopic bdles" are isomorphic.

<u>Corollary 1</u>: If $f_0, f_1 : X \to Y$ are homotopic and $\begin{smallmatrix} E \\ \downarrow \\ Y \end{smallmatrix}$ is a principal G-bdle, then there is an isomorphism

$$\begin{tikzcd} f_0^*E \arrow[rr, "\cong"] \arrow[dr] & & f_1^*E \arrow[dl] \\ & X & \end{tikzcd}.$$

<u>Pf</u>: Any homotopy $H: X\times I \to Y$ $(H_0 = f_0,\ H_1 = f_1)$ induces a bundle htpy $\begin{smallmatrix} H^*E \\ \downarrow \\ X\times I \end{smallmatrix}$ connecting f_0^*E to f_1^*E. $\square$

This corollary tells us that we have a well-defined factorization

$$\begin{tikzcd} f \arrow[r, mapsto] & f^*E \\ \mathrm{Map}(X,B) \arrow[rr] \arrow[dr] & & \mathrm{Prin}_G(X) \\ & {[X,B]} \arrow[ur] & \end{tikzcd}$$

for <u>any</u> paracpt Hausd. space X and <u>any</u> bdle $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$.

<u>Corollary 2</u>: If $F: X \xrightarrow{\simeq} Y$ is a htpy equiv. b/w paracpt. Hausd. spaces, then F induces a <u>bijection</u> $f^*: \mathrm{Prin}_G Y \to \mathrm{Prin}_G X$.

Pf: If $g: Y \to X$ is a htpy inverse to f, then $f^* g^* \simeq \mathrm{Id}_Y$, $g^* f^* \simeq \mathrm{Id}_X$. So $g^* f^* (E) \cong (\mathrm{Id}_X)^* E = E$ and $f^* g^* E \cong (\mathrm{Id}_Y)^* E = E$. $\square$

In particular, this shows that all bdles over a contractible (paracpt., Hausdo.) space are trivial; we used this already in the case where the base is a disk.

We can now complete the proof of our theorem on universal bdles:

Proof: We want to show that $\mathrm{Map}(X, B) \longrightarrow \mathrm{Prin}_G(B)$, $f \longmapsto f^* E$ induces a bijection $[X, B] \xrightarrow{\cong} \mathrm{Prin}_G(B)$ when X is CW and $\pi_* E = 0$. This map is well-defined by the BHT, [Note: every CW cplx is paracompact, by a thm of Miyazaki] and we've proven that it's surjective.

To prove injectivity, we must show that if $f^* E \cong g^* E$ for some $f, g: X \to B$, then $f \simeq g$. This is just a "relative" form of the surjectivity statement, and we prove it the same way: letting $P = f^* E$, we have diagrams
$$\begin{array}{ccc} P & \xrightarrow{\tilde f} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B \end{array}, \quad \begin{array}{ccc} P & \xrightarrow{\tilde g} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & B \end{array}$$
which (by the Key Lemma) correspond to sections s_f, s_g of $P \times_G E$.

These give us a partially defined section of $\begin{array}{c} P\times_G(E\times I) \\ \downarrow \\ X\times I \end{array}$ (defined on $X\times\{0,1\}$) and just as before we can extend this section over the rest of the CW cplx $X\times I$. This gives a section $s: X\times I \longrightarrow P\times_G(E\times I)$, $(x,t)\longmapsto [\tilde{x}, e, t]$ which translates under the Key Lemma to a map $F: X\times I \longrightarrow B\times I$, $(x,t)\longmapsto [s_x(t)]$, and the composite $X\times I \xrightarrow{F} B\times I \longrightarrow B$ is the desired htpy b/w $f,g: X\to B$. □

Porism: If $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ is a principal G-bdle w/ $\pi_k E = 0$ for $k\le n$, then $[X,B] \longrightarrow \mathrm{Prin}_G(X)$, $f \longmapsto f^*E$ is bijective for CW cplx's X of dim'n $< n$, surjective for $\dim X = n$.
(We call E "$(n-1)$-universal".)

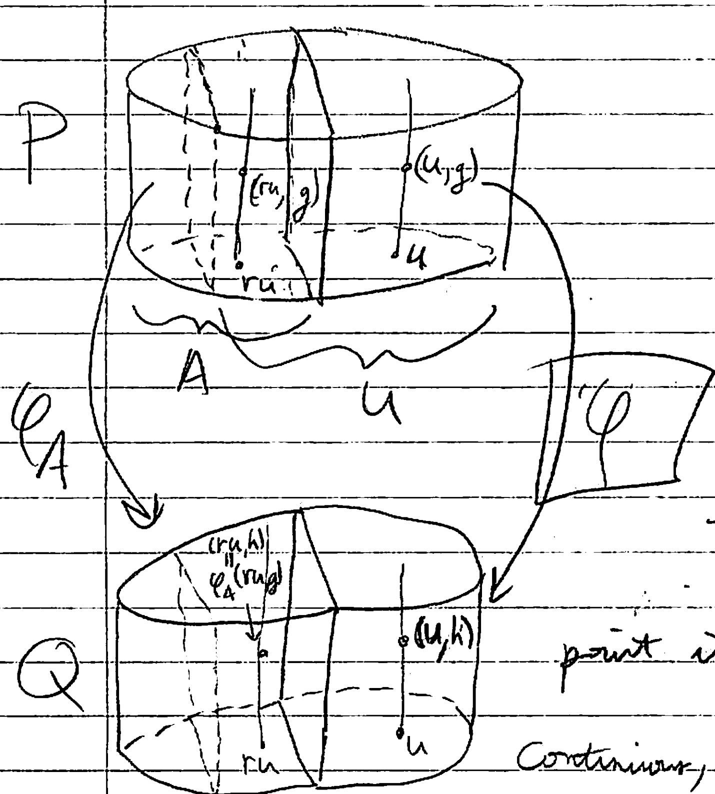
Pf: Just follow the previous proof, noting that $\dim(X\times I) = n+1$, so we don't get injectivity if $\pi_{n+1}E \neq 0$. □

To prove the BHT, we need two lemmas:

Lemma 1: Consider a subset $A\subseteq X$. If $P \searrow_X \swarrow Q$ are principal G-bdles over X, which are both trivial over some open set $U \supset \overline{X-A}$ ("closure") that retracts to $U\cap A$. Then any isomorphism $P|_A \underset{\varphi_A}{\overset{\cong}{\longrightarrow}} Q|_A$ over A extends to an isomorphism $P \underset{\varphi}{\overset{\cong}{\longrightarrow}} Q$ over X.

Pf: To simplify notation, we'll identify $P|_U$ and $Q|_U$ with $U\times G$.

We extend φ_A using the diagram



So we define

$$\varphi(u,g) = (u, \pi_2(\varphi_A(ru,g)))$$

where $\pi_2 : \underset{\substack{\| \\ Q|_U}}{U \times G} \to G$.

This agrees with φ_A on any point in $Q|_A$. To see that φ is continuous, just note that since $U \supset \overline{X - A}$ we have $X = \text{int}(A) \cup U$, and $P = P|_{\text{int}(A)} \cup P|_U$. So we have defined φ on two open sets, and our def'ns agree on the overlap. $\square$

Lemma 2: For any principal G-bdl $\begin{smallmatrix} P \\ \downarrow \\ B \times I \end{smallmatrix}$ and any pt. $b \in B$, $\exists$ an open nbhd $U_b \ni b$ s.t. $P|_{U_b \times I} \cong U_b \times I \times G$.

Pf: For each $t \in I$, $\exists$ a nbhd $U_{b,t} \times (t - \varepsilon_t, t + \varepsilon_t)$ of $B \times I$ over which P is trivial. By cptness of I, $\exists\, t_1, \dots, t_n \in I$ s.t. $I = \bigcup_{i=1}^n (t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i})$. We set $U_b = \bigcap_{i=1}^n U_{b,t_i}$, so that

P is trivial over $U_b \times [0, s_1]$, $U_b \times (s_1 - \varepsilon, s_2]$, ..., $U_b \times (s_n - \varepsilon, 1]$ for some $0 < s_1 < \cdots < s_n$ (the s_i are easily chosen in terms of the t_i). Now Lemma 1 can be applied $n-1$ times, b/c $U_b \times (s_i - \varepsilon, s_{i+1}]$ retracts to $U_b \times (s_i - \varepsilon, s_i]$. $\square$

Proof of the BHT: Given $\begin{matrix} P \\ \downarrow \\ B \times I \end{matrix}$, we need to produce an isom. $P \cong P_0 \times I$. Let $\{U_b\}_{b \in B}$ be the cover guaranteed by Lemma 2, so $P|_{U_b \times I} \cong U_b \times I \times G$. Since B is paracpt and Hausdorff, we can find a locally finite open refinement $\{V_i\}_{i \in I}$ of $\{U_b\}_{b \in B}$ and a partition of unity $\{p_i\}_{i \in I}$ subordinate to $\{V_i\}$ (i.e. $\sum_{i \in I} p_i(b) = 1$ for each $b \in B$, and $\mathrm{supp}(p_i) = \overline{\{b \in B \mid p_i(b) > 0\}} \subseteq V_i$).

If we choose a well-ordering of I, then we can define $W_0 = B \times \{0\}$, and $W_i = \{(b, t) \mid t \leq \sum_{j \leq i} p_j(b)\}$ (for each $i \in I$).

We now proceed by transfinite induction over I.

For each $i \in I$, consider the statement:

$S(i)$: Given any sequence of compatible isom's $P|_{W_j} \xrightarrow[\varphi_j]{\cong} P_0 \times I|_{W_j}$ for $j < i$, there exists an isom. $P|_{W_i} \xrightarrow[\varphi_i]{\cong} P_0 \times I|_{W_i}$ s.t. $\varphi_i|_{W_j} = \varphi_j$ for $j<i$. Moreover, such a comp. seq. exist.

We prove that $S(i)$ holds for all $i \in I$ by transfin. induction.

$S(0)$ is trivially true. Assuming $S(j)$ for all $j<i$, we must prove $S(i)$.

We claim that there is an isom $P|_{\bigcup_{j<i} W_j} \xrightarrow[\varphi]{\cong} P_0 \times I|_{\bigcup_{j<i} W_j}$. To define φ, we use the axiom of choice: for every collection of comp. isom's $\{f_A : P|_A \cong P_0 \times I|_A\}_{A \in \mathcal{S}}$ (where $\mathcal{S}$ is any collection of subsets of $B \times I$) for which there exists an extn $P|_{\bigcup_{A\in\mathcal{S}} A} \cong P_0 \times I|_{\bigcup_{A\in\mathcal{S}} A}$, we choose such an extn. Then by our induction hypothesis, we have chosen isom's $P|_{W_j} \xrightarrow[\varphi_j]{\cong} P_0 \times I|_{W_j}$, $j<i$ which restrict to the identity on $W_0 = B \times \{0\}$ and satisfy $\varphi_l|_{W_k} = \varphi_k$ for $k<l$. Now $P|_{\bigcup_{j<i} W_j} \xrightarrow[\cong]{\cup \varphi_j} P_0 \times I|_{\bigcup_{j<i} W_j}$.

Using Lemma 1, we will extend $\varphi = \bigcup_{j<i} \varphi_j$ to all of W_i.

We have a retraction
$$W_i \longrightarrow \bigcup_{j<i} W_j$$
$$(b,t) \longmapsto (b, \min(t, \textstyle\sum_{j<i} p_j(b)))$$

In addition,

$$W_i - \bigcup_{j<i} W_j = \{(b,t) \mid \sum_{j<i} p_j(b) < t \le \sum_{j\le i} p_j(b)\} \subseteq (\mathrm{supp}(p_i)\times I)\cap W_i$$
$$\subseteq (V_i \times I)\cap W_i .$$

The set $(V_i \times I)\cap W_i$ is an open subset of W_i on which both $P_0 \times I$ and P are trivial, and it contains $\overline{W_i - \bigcup_{j<i} W_j}$ b/c it contains the closed set $(\mathrm{supp}(p_i)\times I)\times W_i$.

By Lemma 2, we can extend the isom. $P_0 \times I|_{\bigcup_{j<i} W_j} \xrightarrow[\cong]{\varphi} P|_{\bigcup_{j<i} W_j}$ to an isom $P_0 \times I|_{W_i} \cong P|_{W_i}$, and moreover we can do the same for any compatible family of isom's $\varphi_j : P|_{W_j} \xrightarrow{\cong} P_0 \times I|_{W_j}$ $(j<i)$. $\square$

<u>Remark</u>: Milnor shows, by point-set arguments, that transfinite induction can be avoided: by Lemma 5.9 in MS, there exists a countable cover $\{U_i\}_{i=1}^{\infty}$ of $B \times I$ over which P is trivial. The process in Lemma 1 then produces a new cover of B consisting of finite intersections of U_i's; hence this new cover is still countable. [Note: one needs here the easy fact that B paracpt $\Rightarrow$ $B \times I$ paracpt.]