Note that if we express v in an o.n. basis $\{e_i\}$, then
"orthonormal"

$$\mu(v) = \langle \Sigma \lambda_i e_i , \Sigma \lambda_i e_i \rangle = \sum_{i,j} \lambda_i \lambda_j ,$$

and the functions $v \mapsto \lambda_i$ are all linear.

Now continuity of $\langle , \rangle : E \times_B E \to \mathbb{R}$ is equivalent to continuity of the associated $\mu : E \to \mathbb{R}$.

Lecture 3 A Euclidean bundle is not only locally isomorphic to a trivial bundle, it is also automatically isometric to a trivial bundle with its standard inner product. This is Lemma 2.4 in MS, and follows from continuity of the Gram-Schmidt orthogonalization process. 1

Clutching Functions for Euclidean Bundles:

Proposition: Let $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ be a Euclidean v. bdle. Then there exist local trivializations $\varphi_i : U_i \times \mathbb{R}^n \to E$ such that the associated clutching functions $\varphi_{ij} : U_i \cap U_j \to GL_n\mathbb{R}$ all land inside $O(n) = \{ A \in GL_n\mathbb{R} \mid A A^T = I_n \}$.

Proof: Simply choose φ_i to be the local isometries guaranteed by Lemma 2.4 (MS). $\square$

This proposition allows us to associate an $O(n)$-bdle to each Euclidean v. bdle, just as we associated a $GL_n\mathbb{R}$-bdle to each ordinary v. bdle. Again, this bdle depends only on the isometry type of the Euclidean bdle.

In the other direction, the mixed bdle

$$\begin{array}{c} P \times_{O(n)} \mathbb{R}^n \\ \downarrow \\ B \end{array} \quad \text{associated to an } O(n)\text{-bdle} \begin{array}{c} P \\ \downarrow \\ B \end{array}$$

inherits a metric from $\mathbb{R}^n$, because the transition fcns are isometries.

Remark: All of this works equally well with $\mathbb{R}$ replaced by $\mathbb{C}$. Metrics on complex bundles are required to be Hermitian, that is they are conjugate-linear in the 2nd coord: $\langle v, zw\rangle = \bar{z}\langle v, w\rangle$ for $z \in \mathbb{C}$.

The transition fcns then lie in $U(n) = \{A \in GL_n\mathbb{C} \mid A\bar{A}^T = I_n\}$, so cplx Hermitian bdles correspond to (principal) $U(n)$-bundles.

Principal Bundles and their Homotopy Theory:

The $GL_n\mathbb{R}/O(n)$ bdles we have associated to vector / Euclidean bdles are examples of the general notion of principal bundles:

<u>Def'n</u> Let G be a topological group (i.e. the mult'n map $G\times G \to G$ and inversion $G \to G$ are continuous).

A (right) principal G-bdle over a space B is a map

$$\begin{array}{c} P \\ \downarrow p \\ B \end{array}$$

together with an open covering $\{U_i\}_{i\in I}$ of B with the following properties:

1) (Local triviality) For each $i \in I$, there is a homeomorphism

$$\varphi_i : U_i \times G \xrightarrow{\cong} p^{-1}(U_i)$$

satisfying $p\varphi_i(u,g) = u$;

2) (Principality) Whenever $U_i \cap U_j \neq \emptyset$, the composite

$$\varphi_{ji} = \varphi_j^{-1}\varphi_i : U_i \cap U_j \times G \longrightarrow U_i \cap U_j \times G$$

has the form $\varphi_{ji}(u,g) = (u, \tau_{ji}(u)g)$ for some $\tau_{ji}(u) \in G$

[here $\tau_{ji}(u)$ depends only on u, not on g].

<u>Remark</u>: The function $u \mapsto \tau_{ji}(u)$ is automatically continuous, b/c $\tau_{ji}(u) = \pi_2(\varphi_{ji}(u,g)) \cdot (\pi_2(u,g))^{-1}$ (where $\pi_2 : U_i \cap U_j \to G$ is projection onto G).

The reason for calling this a <u>right</u> principal bdle is:

<u>Lemma</u>: If $\begin{array}{c} P \\ \downarrow p \\ B \end{array}$ is a (right) principal G-bdle, then P admits a continuous <u>right</u> action $P \times G \to P$ such that

1) The quotient P/G is homeomorphic to B
2) The trivializations $\varphi_i : U_i \times G \to p^{-1}(U_i)$ are G-equivariant (where $(u,g)\cdot h := (u, gh)$).

[Note that 2) implies that G acts freely, and acts transitively on each fiber $p^{-1}(b)$.]

<u>Proof</u>: We transport the action
$$\begin{aligned}(U_i \times G) \times G &\longrightarrow U_i \times G \\ (u_i, g), h &\longmapsto u_i, gh\end{aligned}$$
to P using the local trivializations $\varphi_i : U_i \times G \xrightarrow{\cong} p^{-1}(U_i)$:

For $\in P$, $g \in G$ we define
$$x \cdot g = \varphi_i\left(\varphi_i^{-1}(x) \cdot g\right).$$

This is well-defined by principality: if $u = p(x) \in U_i \cap U_j$, we must check that $\varphi_i(\varphi_i^{-1}(x)\cdot g) = \varphi_j(\varphi_j^{-1}(x)\cdot g)$, i.e. that $\varphi_j^{-1}\varphi_i(\varphi_i^{-1}(x)\cdot g) = \varphi_j^{-1}(x)\cdot g$.

Letting $\varphi_i^{-1}(x) = (u,h)$, we have
$$\begin{aligned}\varphi_j^{-1}\varphi_i(\varphi_i^{-1}(x)\cdot g) &= \varphi_{ji}((u,h)\cdot g) = \varphi_{ji}(u, hg) \\ &= (u, \tau_i(u) hg) = (u, \tau_i(u) h)\cdot g \\ &= \varphi_{ji}\,(u,h)\cdot g = \varphi_j^{-1}(\varphi_i(u,h))\cdot g \\ &= \varphi_j^{-1}(x)\cdot g.\end{aligned}$$

To see that $P/G \cong B$, note that we have a comm. diagram
$$\begin{tikzcd} & P \arrow[dl, "q"'] \arrow[d, "p"] \\ P/G \arrow[r, dashed, "f"] & B \end{tikzcd}$$
in which f is a continuous bijection. To see that f is an open map, consider any open set $\bar{V} \subseteq P/G$. Then $V = q^{-1}(\bar{V})$ is open in P, and $f(\bar{V}) = p(q^{-1}\bar{V})$. But p is an open map, b/c locally it is just the projection $U_i \times G \to U_i$. $\square$

Basic Examples:

The $GL_n\mathbb{R}/O(n)$-bdle associated to a vector/Euclidean bdle are principal bundles. In fact, for any group G and any clutching data $\varphi_{ji} : U_i \cap U_j \to G$ ($\{U_i\}_i$ an open cover of some base B), the bundle
$$\begin{array}{c} P = \left(\coprod_i U_i \times G\right) \Big/ (u, g) \sim (u, \varphi_{ji}(u) g) \\ \downarrow p \\ B \end{array}$$
is principal, w/ local trivializations the inclusions $U_i \times G \hookrightarrow P$. The associated action is just $[u, g] \cdot h = [u, gh]$.

[The fact that $U_i \times G \hookrightarrow P$ is a homeomorphism onto $p^{-1}(U_i)$ follows from the fact that $\begin{array}{ccc} U_i \cap U_j \times G & \to & U_i \cap U_j \times G \\ u \quad g & \mapsto & (u, \varphi_{ji}(u) g) \end{array}$ is a homeomorphism.]

<u>Note</u>: The fact that $U_i \times G \hookrightarrow P$ is a homeomorphism onto its image $p^{-1}(U_i)$ follows from the fact that

$$\begin{array}{ccc} U_i \cap U_j \times G & \longrightarrow & U_i \cap U_j \times G \\ (u, g) & \longmapsto & (u, \varphi_{ji}(u) g) \end{array}$$ is a homeomorphism.

Maps b/w Principal Bdles:

<u>Def'n</u>: If $\begin{array}{c} P_1 \\ \downarrow p_1 \\ B_1 \end{array}$ and $\begin{array}{c} P_2 \\ \downarrow p_2 \\ B_2 \end{array}$ are principal G-bdles, a map from $P_1 \to P_2$ is a diagram

$$\begin{tikzcd} P_1 \arrow[r, "\varphi"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowright"] & P_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "\bar{\varphi}"] & B_2 \end{tikzcd}$$

indicates commutativity

<u>in which φ is</u> G-equivariant.

We've seen that fiber-wise isomorphisms of V. bdles are honest isomorphisms; here is the analogue for principal bdles.

<u>Prop'n</u>: If $\begin{tikzcd} P_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & P_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$ is a map of principal G-bdles (covering Id_B) then φ is a homeomorphism (and its inverse $\varphi^{-1}: P_2 \to P_1$ is also a map of principal bdles).

<u>PF</u>: Locally, φ has the form $\begin{array}{ccc} U \times G & \xrightarrow{\varphi} & U \times G \\ (u, g) & \longmapsto & (u, \varphi_2(u,g)) \end{array}$, where $g \mapsto \varphi_2(u,g)$ is G-equivariant (wrt right mult. in G). This means $\varphi_2(u,g) = h(u) g$

where $h(u) := \varphi_2(u,g)g^{-1}$. Hence $h: U \to G$ is continuous, and now $\varphi^{-1}: U\times G \to U\times G$ is the continuous map $(u,g)\mapsto (u, h(u)^{-1}g)$.

So φ is a continuous bijection, and its inverse is continuous. $\square$

Corollary: If $\begin{smallmatrix} P \\ \downarrow p \\ B\end{smallmatrix}$ is a principal G-bdle admitting a continuous section $\begin{tikzcd} P \arrow[d, shift right, "p"'] \\ B \arrow[u, shift right, "s"'] \end{tikzcd}$ $(ps = 1d_B)$ then P is trivial, i.e. there is an isom. of G-bdles $\begin{tikzcd} P \arrow[rr, "\cong"] \arrow[dr] & & B\times G \arrow[dl] \\ & B & \end{tikzcd}$.

Pf: Define $\varphi: \begin{matrix} B\times G \longrightarrow P \\ (b,g) \longmapsto s(b)\cdot g \end{matrix}$, and apply the Prop'n. $\square$

Here is another application of the Prop'n:

Exercise: Say $\begin{smallmatrix} V \\ \downarrow \\ B\end{smallmatrix}$ is a v. bdle, and say $\{U_i, \varphi_i\}_i$, $\{V_j, \psi_j\}_j$ are two different local trivializations of V. Then the associated principal $GL_n\mathbb{R}$ bdles for these different clutching data are isomorphic.

Pullbacks:

Given a map $F: X\to Y$ and a bdle (v. bdle, Euclidean bdle, principal G-bdle) $\begin{smallmatrix} E \\ \downarrow \pi \\ B\end{smallmatrix}$, the pullback $F^*E = \{(x,e)\in X\times E \mid F(x) = \pi(e)\}$ is a bdle over B (of the same type).

[If $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ is trivial over $\{U_i\}$, f^*E will be trivial over $\{f^{-1}(U_i)\}$.]

Our next goal is the following theorem, which describes the set $\mathrm{Prin}_G(X) = \{\text{principal } G\text{-bdles over } X\}/\cong$ of isom. classes of G-bdles <u>homotopically</u>.

<u>Theorem</u>: If $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ is a principal G-bdle such that all htpy groups $\pi_*(E)$ are trivial, then for every CW cplx X, the map

$$\begin{array}{ccc} \mathrm{Map}(X, B) & \xrightarrow{\Phi} & \mathrm{Prin}_G(X) \\ f: X \to B & \longmapsto & [f^*(E)] \end{array}$$

factors through homotopy classes and gives a <u>bijection</u>

$$[X, B] \xrightarrow{\cong} \mathrm{Prin}_G(X).$$

<u>Notation/Terminology</u>: The bdle $E \to B$ is called a <u>universal</u> principal G-bdle. One often denotes the base space B by BG and the total space E by EG; BG is called a <u>classifying space</u> for G.

The proof of this theorem will require several important ideas, constructions and results.

We begin by considering surjectivity of Φ.

<u>Lemma</u>: Any map of principal G-bdles
$$\begin{tikzcd} P_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "p_1"] & P_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "\varphi"] & B_2 \end{tikzcd}$$
induces an <u>isomorphism</u> $P_1 \xrightarrow{\cong} \varphi^* P_2$.

<u>Pf</u>: We have a map $\begin{array}{l} P_1 \longrightarrow \varphi^* P_2 \\ x \longmapsto (p_1(x), \tilde{\varphi}(x)) \end{array}$ which is equivariant and covers Id_{B_1}. The result now follows from the Prop'n. $\square$

To prove that every bdle $\begin{array}{c} P \\ \downarrow \\ X \end{array}$ over a CW cplx is pulled back from the universal bdle $\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$, we just need to construct an equivariant map $P \xrightarrow{\tilde{\varphi}} EG$. Since $X = P/G$, the diagram
$$\begin{tikzcd} P \arrow[r, "\tilde{\varphi}"] \arrow[d] & EG \arrow[d] \\ X = P/G \arrow[r, dashed, "\varphi"] & BG = EG/G \end{tikzcd}$$
can always be filled in.

<u>Key Lemma</u> (Ehresman): Given a principal G-bdle $\begin{array}{c} P \\ \downarrow p \\ X \end{array}$ and a G-space E, there is a bijection between G-equivariant maps $P \to E$ and sections of the mixed bundle
$$\begin{array}{c} P \times_G E = (P \times E)/_{(p,e)\sim(pg,eg)} \\ \downarrow \\ X \end{array}.$$

<u>Proof</u>: Given a G-map $P \xrightarrow{\varphi} E$, we have a diagram:
$$\begin{tikzcd} P \arrow[r, "(Id,\varphi)"] \arrow[d] & P \times E \arrow[d, "\pi"] \\ X = P/G \arrow[r, dashed, "S"] & P \times_G E \end{tikzcd}.$$
Equivariance of φ implies that $\pi \circ (Id, \varphi)$ factors through P/G, and S is the desired section.

In the other direction, given $\begin{array}{c} P \times_G E \\ s \uparrow \downarrow \pi \\ P/G = X \end{array}$, we define $P \xrightarrow{\varphi} E$ as follows:

We can always write $s([p]) = [p, e]$ for some (unique) $e \in E$ and we set $\varphi(p) = e$. Continuity of φ can be checked locally, using the fact that

$$\begin{aligned} (U \times G) \times_G E &\xrightarrow{\cong} U \times E . \quad \square \\ [u, g, e] &\longmapsto (u, eg^{-1}) \\ [u, 1, e] &\longleftarrow\!\shortmid (u, e) \end{aligned}$$

Lecture 4:

Prop'n: Let X be a CW cplx, and let $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ be a fiber bundle with fiber F. If $\pi_* F = 0$ for $* \leq \dim X$ then E admits a section (if $\dim X = \infty$, we just require $\pi_* F = 0$ for $* \geq 0$).

In particular, the mixed bdles $\begin{array}{c} P \times_G EG \\ \downarrow \\ X \end{array}$ always admit sections, so we obtain the required G-map $P \to EG$.

In the proof, we will need to use the fact that every principal G-bdle over a contractible space is trivial. This follows from the Bundle Htpy Theorem, which we'll prove next time.