The Euler Class as an Obstruction

Thm: If $\begin{matrix}\xi^n\\ \downarrow\\ B\end{matrix}$ is an oriented n-plane bdle over a CW cplx, then ξ^n admits a nowhere-zero section $\Longleftrightarrow e(\xi^n)=0$.

In particular, ~~a cm~~ an oriented mfld M admits a nowhere vanishing v.field $\Longleftrightarrow e(TM)=0$ $\Longleftrightarrow \langle e(TM),[M]\rangle = \chi(M)=0$.

The idea behind this result is that there is a natural way to define an integral cohomology class which serves as the obstruction to the existence of a section. We'll then need to compare it with the Euler class.

First Observation: ξ admits a section ~~iff~~ $\Longleftrightarrow \xi\cong\xi'\oplus\varepsilon^1 \Longleftrightarrow$ classifying map of ξ lifts from $B\cancel{S}O(n)$ to $B\cancel{S}O(n-1)$

[we could work w/ $SO(n)$, but that's not strictly necessary].

$$\begin{tikzcd} & BO(n-1) \arrow[d, "\leftarrow \text{ classifies } \gamma_n\oplus\varepsilon^1"] \\ B \arrow[ur, dashed] \arrow[r, "\text{classifies } \xi"'] & BO(n) \end{tikzcd}$$

So we need to examine when such a lift exists.

To study this, we'll use a convenient model for $BO(n-1)$: $EO(n)$ = univ princ $O(n)$-bdle, $O(n-1) \subseteq O(n)$ acts on $EO(n)$ and

$$\begin{array}{c} EO(n) \\ \downarrow \\ EO(n)/O(n-1) \end{array}$$

is a univ. princ $O(n-1)$-bdle.

Now have fib'n sequence

$$\begin{array}{c} O(n)/O(n-1) \\ \downarrow \\ EO(n)/O(n-1) = BO(n-1) \\ \downarrow \\ EO(n)/O(n) = BO(n) \end{array}$$

The fiber $O(n)/O(n-1)$ is just S^{n-1}, b/c $O(n)$ acts transitively on S^{n-1} w/ stabilizer $O(n-1)$.

So our lifting problem becomes

$$\begin{tikzcd} & S^{n-1} \arrow[d] \\ & BO(n-1) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & BO(n) \end{tikzcd}$$

Say $B = B' \cup e_n$ w/ B' n-dim'l (or less). If we assume $\exists$ a lift over B' then we just want to extend it over e_n, i.e. we want to extend a map from $\partial e_n = S^{n-1}$ to e_n:

$$\begin{tikzcd} & B' \arrow[r] \arrow[dd] & BO(n-1) \arrow[dd] \\ \partial e_n \arrow[ur, hook] \arrow[urr] \arrow[dr, hook] & & \\ & B' \cup e_n \arrow[r] \arrow[uur, dashed, "\exists ?"] & BO(n) \end{tikzcd}$$

The map $\partial e_n = S^{n-1} \to BO(n-1)$ lies in $\pi_{n-1}(BO(n-1))$, but in fact it lies in

$$\ker(\pi_{n-1} BO(n-1) \longrightarrow \pi_{n-1} BO(n))$$

b/c we have an ext'n $e_n \longrightarrow BO(n)$.

In fact, this ext'n gives a specific null htpy of the map $\partial e_n \to BO(n-1) \to BO(n)$, and using the htpy lifting property we obtain a specific element in $\pi_{n-1}(\text{fiber}) = \pi_{n-1}(S^{n-1}) = \mathbb{Z}$:

$$\begin{tikzcd} S^{n-1}\times 0 = \partial e_n \arrow[r] \arrow[d, hook] & BO(n-1) \arrow[d] \\ S^{n-1}\times I \arrow[r] \arrow[ur, dashed, "\tilde{H}"] & e_n \arrow[r] & BO(n) \end{tikzcd}$$

$$\tilde{H}_1 : S^{n-1} \to \text{fiber}(BO(n-1) \to BO(n)) = S^{n-1}$$

This forces one to use "local coeff's" in general

One needs to ~~keep~~ track of basepts, it's just like our construction of the LES in Lecture 6.

[Technically, these isom's are not canonical although they would be if we worked w/ the univ. oriented bdle $\begin{matrix} ESO(n) \\ \downarrow \\ BSO(n) \end{matrix}$.]

The assignment

$$e_n \longmapsto [\partial e_n] \in \ker \cong \mathbb{Z}$$

is an integral n-cocycle in the cellular chain cplx.

Fact: As we vary our original lift $B' \to BO(n-1)$ on the $(n-1)$-cells, this cocycle varies through its entire cohomology class (precisely).
Hence we have a well-defined cohomology class, which vanishes $\Longleftrightarrow$ the lift extends over all of B.

This is proven in Steenrod's Fibre Bundles, §31–34.

Note that a lift always exists over the (n-1)-skeleton of B: inductively, we have

$$\begin{tikzcd} K \arrow[r] \arrow[d] & BO(n-1) \arrow[d] \\ K \cup e_k \arrow[r] \arrow[ur, dashed, "\exists ?"] & BO(n) \end{tikzcd} \qquad \partial e_k \hookrightarrow K, \quad \partial e_k \to BO(n-1), \quad e_k \hookrightarrow K \cup e_k$$

We want to know whether $[\partial e_k] \in \pi_{k-1} BO(n-1)$ ~~…~~ is zero. It certainly maps to zero in $\pi_{k-1} BO(n)$, and

$$\pi_{k-1} S^{n-1} \to \pi_{k-1} BO(n-1) \longrightarrow \pi_{k-1} BO(n-1) \to \pi_{k-2} S^{n-1}$$

shows that $\longrightarrow$ is an isom. if $k-1 < n-1$ i.e. if $k < n$.

Hence over the (n-1)-skeleton of B, ξ admits a section, and we have a well-defined obstruction to extending that section over all of $B^{(n)}$.

<u>Thm:</u> This obstruction class is precisely the Euler class if ξ is oriented.

<u>Sketch of Pf:</u> Consider

$$\begin{tikzcd} \pi_0^* \xi \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \xi \arrow[d] \\ \xi_0 \arrow[r] & B \end{tikzcd}$$

. There is a nowhere-zero section of $\pi_0^* \xi$:

$$\vec{v} \longmapsto (\vec{v}, \vec{v})$$

for $\vec{v} \in \xi_0$ (i.e. $\vec{v} \neq 0$).

So the obstruction class $O(\pi_0^* \xi)$ ~~…~~ $= 0$

This class is natural, i.e. $O(\pi_0^* \xi) = \pi_0^*(O(\xi))$.

Recall that we have the Gysin sequence:

$$\begin{array}{ccccc} H^0(B) & \xrightarrow{\cup e(\xi)} & H^n(B) & \xrightarrow{\pi_0^*} & H^n(E_0) \\ & & O(\xi) & \longmapsto & 0 \end{array}$$

so $O(\xi) = \lambda \cup e(\xi)$ for some $\lambda \in H^0 B$.

Note that λ is a universal constant, b/c its value on $\xi = \gamma_n$ determines its values everywhere
$$\begin{array}{c}\gamma_n \\ \downarrow \\ Gr_n \mathbb{C}^\infty\end{array}$$
else.

So $e(\xi)$ and $O(\xi)$ agree up to a (non-zero) scalar multiple, and some calculations w/ Euler char. show that $\lambda = 1$. [See MS 12.5].

In any event, $e(\xi) = 0 \iff O(\xi) = 0$, so if $e(\xi) = 0$ then ξ admits a section. $\square$

Stiefel-Whitney Classes and Obstructions

We can ask various more general questions about existence of sections: the most interesting (or at least the one w/ the easiest answer) is: When does $\begin{array}{c}\xi^n \\ \downarrow \\ B\end{array}$ admit k linearly indep sections over ~~the~~ $B^{(n-k+1)}$

This question is controlled by a "priming obstruction" lying in $H^{n-k+1}(B; \pi_{n-k}(\underset{\sim}{V_k}F))$

~~bdle~~ k frames in a fiber F of ξ.

The groups $\pi_{n-k}(V_k F)$ are non-canonically isom. to $\mathbb{Z}$ when $n-k$ is odd and isom. to $\mathbb{Z}/2$ when $n-k$ is even.

Again, there always exist k lin. indep. sections over the $n-k$ skeleton, b/c the obstructing htpy gp vanishes.

~~The class is~~

These classes reduce to w_{n-k+1} when the coeffs are reduced to $\mathbb{Z}/2$.

Again, universal arguments and a specific computation (over $\mathbb{R}P^n$, this time) can be used to prove this.

Further comments $e\left(\begin{smallmatrix}\gamma_n \\ \downarrow \\ Gr_n\mathbb{C}^\infty\end{smallmatrix}\right) = \pm c_n$:

- agree up to integer mult. b/c both are in ker $(H^* Gr_n\mathbb{C}^\infty \longrightarrow H^* Gr_{n-1}\mathbb{C}^\infty)$
- $\exists$ complex $2n$-mflds w/ $\chi = 2n+1 \Longrightarrow e(\gamma_{2n}) = \lambda c_{2n}$ w/ $\lambda \mid 2n+1$
- $(Gr_{2n})^2 \to Gr_{2n}\mathbb{C}^\infty$

$\gamma_n \oplus \gamma_n \longleftarrow\!\shortmid \gamma_{2n}$

$(c_n)^2 \longleftarrow\!\shortmid c_{2n}$ $\quad e_{2n} = c_n$

$\lambda e_{2n}^2 \longleftarrow\!\shortmid e_{4n}$

Need to do a computation to show $\lambda = \pm 1$

<u>Orientations of Hypersurfaces:</u>

If $H^n \subseteq \mathbb{R}^{n+1}$ is a hypersurface, then

H is orientable $\Longleftrightarrow$ normal bdle is trivial, i.e.
$\exists$ well-defined "outward normal vector"

<u>Pf</u>: H or. $\Leftrightarrow$ TH or. $\Longleftrightarrow$ $w_1 TH = 0$

$$w_1(TH) + w_1(\underbrace{NH}_{\text{normal bdle}}) = w_1(T\mathbb{R}^{n+1}) = 0$$

so $w_1 NH = 0 \Leftrightarrow w_1 TH = 0$.

And NH or. $\Leftrightarrow$ $w_1 NH = 0$. $\blacksquare$