

## The Euler Class as an Obstruction

Thm: If  $\begin{array}{c} \mathbb{F}^n \\ \downarrow \\ B \end{array}$  is an oriented  $n$ -plane bundle, then  $\mathbb{F}^n$  admits a nowhere-zero section  $\Leftrightarrow e(\mathbb{F}^n) = 0$ .

In particular, an oriented manifold  $M$  admits a nowhere vanishing v. field  $\Leftrightarrow e(TM) = 0$   
 $\Leftrightarrow \langle e(TM), [M] \rangle = \chi(M) = 0$ .

The idea behind this result is that there is a natural way to define an integral cohomology class which serves as the obstruction to the existence of a section.

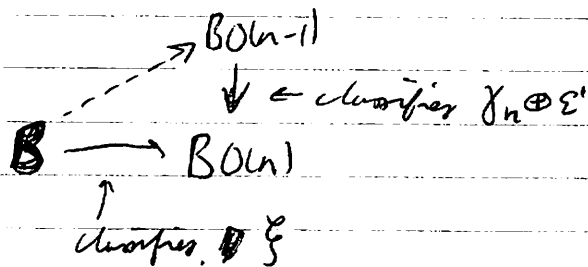
We'll then need to compare it with the Euler class.

First Observation:  $\mathbb{F}$  admits a section  $\Leftrightarrow$

$$\Leftrightarrow \mathbb{F} \cong \mathbb{F}' \oplus \mathbb{F}' \Leftrightarrow$$

classifying map of  $\mathbb{F}$  lifts from  $B\mathbb{O}(n)$  to  $B\mathbb{O}(n-1)$

[we could work w/  $SO(n)$ , but that's not strictly necessary]



So we need to examine when such a lift exists.

To study this, we'll use a convenient model  
for  $BO(n-1)$ :  $EO(n) = \text{univ. princ. } O(n)\text{-bundle}$ ,

$O(n-1) \subseteq O(n)$  acts on  $EO(n)$  and

$EO(n)$

$\downarrow$

$EO(n)/O(n-1)$

is a univ. princ.  $O(n-1)$ -bundle.

Now have fib'n sequence

$O(n)/O(n-1)$

$\downarrow$

$EO(n)/O(n-1) = BO(n-1)$

$\downarrow$

$EO(n)/O(n) = BO(n)$

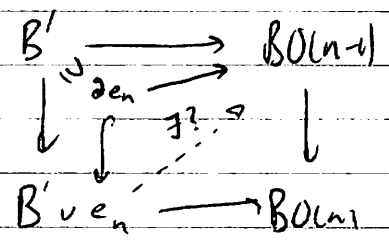
The fiber  $O(n)/O(n-1)$  is just  $S^{n-1}$ , b/c  $O(n)$  acts transitively on  $S^{n-1}$  w/ stabilizer  $O(n-1)$ .

So our lifting problem becomes

$$\begin{array}{ccc} & S^{n-1} & \\ & \downarrow & \\ & BO(n-1) & \\ \dashrightarrow & & \downarrow \\ B & \longrightarrow & BO(n) \end{array}$$

Say  $B = B' \vee e_n$  w/  $B'$   $n$ -dim'l (or less). If we assume

$\exists$  a lift over  $B'$  then we just want to extend it over  $e_n$ ,  
i.e. we want to extend a map from  $\partial e_n = S^{n-1}$  to  $e_n$ :

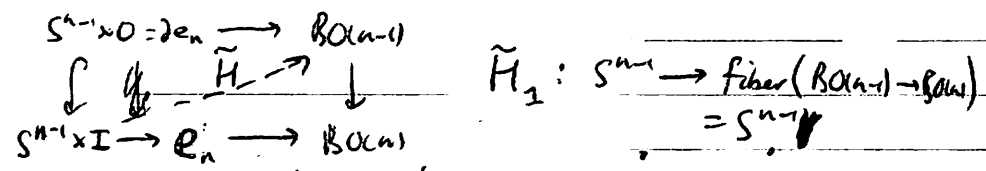


The map  $\partial_n = S^{n-1} \rightarrow BO(n-1)$  lies in  $\pi_{n-1}(BO(n-1))$ , but in fact it lies in

$$\ker(\pi_{n-1} BO(n-1) \rightarrow \pi_{n-1} BO(n))$$

b/c we have an ext'n  $e_n \rightarrow BO(n)$ .

In fact, this ext'n gives a specific null htpy of the map  $\partial_n \rightarrow BO(n-1) \rightarrow BO(n)$ , and using the htpy lifting property, we obtain a specific element in  $\pi_{n-1}(\text{fiber}) \cong \pi_{n-1}(S^{n-1}) = \mathbb{Z}$ :



This forces one to use "local lifts" in general

One needs to keep track of basepts, - ~~but~~ it's just like our construction of the LES in Lecture 6. [Technically, these rooms are not connected although they

would be if we worked w/ the union. overcame this  $\downarrow$   $\mathbb{Z}$  total  $\downarrow$   $BO(n)$ ]

The assignment 
$$e_n \mapsto [\partial_n] \in \text{fiber} \cong \mathbb{Z}$$

is an ~~an~~ integral  $n$ -cocycle in the cellular chain cplx.

Fact: As we vary our original <sup>lift</sup> ~~map~~  $B' \rightarrow BO(n-1)$  on the  $(n-1)$ -cells, this cocycle ~~also~~ varies through its entire cohomology class (precisely). Hence we have a well-defined class, which vanishes  $\iff$  the lift extends over all of  $B$ .

This is proven in Steenrod's *Fibre Bundles*, § 31-34.

Note that a lift always exists over the  $(n-1)$ -skeleton of  $B$ : inductively, we have

$$\begin{array}{ccc} K & \longrightarrow & B O(n-1) \\ \downarrow \cong \partial e_k & \nearrow \beta & \downarrow \\ K \cup e_k & \longrightarrow & B O(n) \end{array}$$

We want to know whether  $[\partial e_k] \in \pi_{k-1} B O(n-1)$

~~is~~ is zero. It certainly maps to zero

in  $\pi_{k-1} B O(n)$ , and

$$\pi_{k-1} S^{n-1} \rightarrow \pi_{k-1} B O(n-1) \rightarrow \pi_{k-1} B O(n) \rightarrow \pi_{k-2} S^{n-1}$$

shows that  $\pi_{k-1} B O(n-1) \rightarrow \pi_{k-1} B O(n)$  is an isomorphism if  $k-1 < n-1$ , i.e.


if  $k < n$ .

Hence over the  $(n-1)$ -skeleton of  $B$ ,

$\mathcal{Q}$  admits a section, and we have a well-defined

obstruction to extending that section over all of  $B^{(n)}$ .

Thm: This obstruction class is precisely the Euler class of  $\mathcal{Q}$  if  $\mathcal{Q}$  is oriented.

Sketch of Pf: 

Consider  $\begin{array}{ccc} \pi_0^* \mathcal{Q} & \rightarrow & \mathcal{Q} \\ \downarrow \Gamma & & \downarrow \\ \mathcal{Q}_0 & \rightarrow & B \end{array}$

There is a nowhere-zero section of  $\pi_0^* \mathcal{Q}$ :

$$\vec{v} \mapsto (\vec{v}, \vec{v})$$

for  $\vec{v} \in \mathcal{Q}_0$  (i.e.  $\vec{v} \neq 0$ ).

So the obstruction class  $O(\xi) \neq 0$

This class is natural, i.e.  $O(\pi_0 \xi) = \pi_0(O(\xi))$ .

Recall that we have the Gysin sequence:

$$\begin{array}{ccccc} H^0(B) & \xrightarrow{u \cup e(\xi)} & H^n(B) & \xrightarrow{\pi_0^*} & H^n(E_0) \\ & & O(\xi) & \longmapsto & 0 \end{array}$$

so  $O(\xi) = \lambda \cup e(\xi)$  for some  $\lambda \in H^0(B)$ .

Note that  $\lambda$  is a universal constant, b/c its value on  $\xi = \gamma_n$  determines its values everywhere

$$\downarrow$$

$\text{Gr}_n \mathbb{C}^\infty$

else.

So  $e(\xi)$  and  $O(\xi)$  agree up to a (non-zero) scalar multiple, and some calculations w/ Euler Char.

Show that  $\lambda = 1$ . [See MS 12.5]

In any event,  $e(\xi) \neq 0 \implies O(\xi) \neq 0$ , so

if  $e(\xi) = 0$  then  $\xi$  admits a section.  $\square$

### Stiefel-Whitney Classes and Obstructions

We can ask various more general questions about existence of sections: the most interesting (or at least the one w/ the easiest answer) is: When does  $\xi^n$  admit  $k$  linearly indep sections over  $B^{(k-1)}$ ?

This question is controlled by a "primary obstruction" lying in  $H^{n-k+1}(B; \pi_{n-k}(V_k F))$   
 $\underbrace{\pi_{n-k}(V_k F)}$  is  $k$  pages in a fiber  $F$  table of §.

The groups  $\pi_{n-k}(V_k F)$  are non-canonically isom to  $\mathbb{Z}$  when  $n-k$  is odd and isom. to  $\mathbb{Z}/2$  when  $n-k$  is even.

Again, there always exist ~~some~~  $k$  lin. indep. vectors over the  $n-k$  skeleton b/c the obstructing  $k$  type of vanishes.

~~The classes in~~

These classes reduce to  $w_{n-k+1}$  when the coeffs are reduced to  $\mathbb{Z}/2$ .

Again, universal arguments and a specific computation (over  $\mathbb{R}P^n$ , this time) can be used to prove this.

Further comments  $e(\begin{smallmatrix} \delta_n \\ \vdots \\ \delta_1 \\ \text{Gr}_n \mathbb{C}^\infty \end{smallmatrix}) = \pm c_n$ :

- Agree up to ~~some~~ integer mult. b/c both are in  $H^*(\text{Gr}_n \mathbb{C}^\infty \rightarrow H^*(\text{Gr}_n \mathbb{C}^\infty))$
- $\int \text{complex } \mathbb{Z}^n$  - implies w/  $\chi = 2^{n+1} \Rightarrow e(\delta_{2n}) = \pm c_{2n}$  w/  $\chi = 2^{n+1}$
- $(\text{Gr}_{2n})^k \rightarrow \text{Gr}_{2n} \mathbb{C}^\infty$

$\delta_n \otimes \delta_n \leftarrow \delta_{2n}$  Need a computation to show  $d = \pm 1$

$(\mathbb{C}^n)^k \leftarrow \mathbb{C}_{2n}^k \rightarrow \mathbb{R}P_{2n} = c_n$   
 $\lambda \mathbb{C}_{2n}^k \leftarrow \mathbb{R}P_{2n}$

## Orientation of Hypersurfaces:

If  $H^n \subseteq \mathbb{R}^{n+1}$  is a hypersurface, then

$H$  is orientable  $\iff$  normal bundle is trivial, i.e.  
 $\exists$  well-defined, outward normal vector.

Pf.  $H$  or  $\iff$   $TH$  or  $\iff$   $\omega_1 TH = 0$

$$\omega_1(TH) + \omega_1(\underbrace{NH}_{\text{normal bundle}}) = \omega_1(T\mathbb{R}^{n+1}) = 0$$

$$\text{or } \omega_1 NH = 0 \iff \omega_1 TH = 0.$$

And  $NH$  or or  $\iff$   $\omega_1 NH = 0$ .  $\square$