Lecture 23 Euler Characteristic + the Euler Class

Last time:

The diagonal embedding

$$M^n \xhookrightarrow{\ \Delta\ } M\times M$$

gives us a "fundamental class"

$$u' \in H^n(M\times M,\ M\times M - \Delta M) \cong H^n(\underbrace{\nu^k}_{\nu^k \cong TM}(\Delta M),\ \nu^k(\Delta M)_0)$$

corresponding to the Thom class of TM.

Def'n: The diagonal cohomology class $u'' \in H^n(M\times M)$ is $u'|_{M\times M}$.

So we have:

$$\boxed{\Delta^*(u'') = e(TM)}$$

b/c

$$\begin{tikzcd}
u'' & u' \arrow[l, mapsto] \\
H^n M\times M \arrow[d, "\Delta^*"] & H^n(M\times M, M\times M-\Delta M) \arrow[l] \arrow[d, phantom, "\| \text{ Excise complement of tub. nbhd}"] \\
H^n M & H^n(TM,(TM)_0) \arrow[l] \\
e_{TM} & U_{TM} \arrow[l, mapsto]
\end{tikzcd}$$

Lemma 11.8 $(a\times 1)\cup u'' = (1\times a)\cup u''$ for any $a \in H^*M$

Pf:

$$\begin{tikzcd}
a\times 1,\ 1\times a \arrow[rr, mapsto] & & (a\times 1)_\varepsilon, (1\times a)_\varepsilon \\
H^i(M\times M) \arrow[rr] \arrow[d, "\cup u'"] \arrow[dd, bend right=60, "\cup u''"'] & & H^i(\underbrace{N_\varepsilon}_{\text{tubular nbhd of } \Delta M}) \arrow[d, "\cup u'|_{(N_\varepsilon, N_\varepsilon - \Delta M)}"] \\
H^{i+n}(M\times M, M\times M-\Delta M) \arrow[rr, "\cong"] \arrow[d] & & H^{i+n}(N_\varepsilon, N_\varepsilon - \Delta M) \\
H^{i+n}(M\times M) & &
\end{tikzcd}$$

Suffices to check that $(a\times 1)_\varepsilon = (1\times a)_\varepsilon$. But

$$\begin{tikzcd}
 & M\times M \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
M & & M
\end{tikzcd}
\qquad a \mapsto a\times 1,\quad a \mapsto 1\times a,$$

and p_1, p_2 are htpic on $N_\varepsilon \simeq \Delta M$. □

<u>Slant Product</u>: We now work (implicitly) w/ field coefficients (and we're mainly interested in $\mathbb{Q}$ or $\mathbb{Z}/2$)

Assume X, Y are finite CW cplxs (e.g. cpt mflds)

We can define a "division" operation

$$H^n(X\times Y)\otimes H_k(Y)\xrightarrow{/} H^{n-k}(X)$$

using the Kunneth Isom $H^n(X\times Y)\cong\bigoplus_{p+q=n}H^pX\otimes H^qY$:

$$\Big(\bigoplus_{p+q=n}H^pX\otimes H^qY\Big)\otimes H_kY\xrightarrow[\text{project}]{}H^{n-k}X\otimes H^kY\otimes H_kY\longrightarrow H^{n-k}X$$

$$a\otimes b\otimes\beta\longmapsto a\langle b,\beta\rangle$$

Kronecker pairing b/w H_kY and $H^kY\cong\mathrm{Hom}(H_kY,\mathbb{Q})$.

For $p\in H^n(X\times Y)$, $\beta\in H_kY$, we write this op'n as $p\otimes\beta\longmapsto p/\beta$. Note that this op'n is <u>zero</u> on any simple tensor $p=a\times b$ with $|b|\neq|\beta|=k$ (where $|\cdot|$ denotes degree of the elt).

In subsequent calculations, this fact should be kept in mind.

By abuse of notation, we set $\langle b,\beta\rangle\overset{\text{def}}{=}0$ if $|b|\neq|\beta|$, so that the formula $(a\otimes b)/\beta=a\langle b,\beta\rangle$ always holds.

<u>Lemma</u> ("left-linearity")

$$[(a\times 1)\cup p]/\beta=a\cup(p/\beta).$$

<u>Pf</u>: By linearity, we can assume $p=q\times b$. Now

$$[(a\times 1)\cup(q\times b)]/\beta=((a\cup q)\times b)/\beta=(a\cup q)\langle b,\beta\rangle=a\cup(q\langle b,\beta\rangle)$$
$$=a\cup[(q\times b)/\beta].\ \square$$

Key Computation:

Lemma 11.9 If M^n is cpt and oriented, then $u'' \in H^n(M \times M)$ satisfies:

$$u''/[M] = 1 \in H^0(M).$$

(So u'' is "dual" to the fundamental homology class of M.)

(Sketch) Pf: Since $u''/[M] \in H^0(M)$, we can just compute its restriction at each point $x \in M$.

Main point: $[M]|_{(M, M-x)} \in H_n(M, M-x)$ is the "orientation generator", and

$$u'' = u'|_{M \times M}, \text{ where } u' \in H^n(M \times M, M \times M - \Delta M)$$

satisfies

$$u'|_{((N_\varepsilon)_x, (N_\varepsilon)_x - \{(x,x)\})} = \text{or. gen. of } (N_\varepsilon)_x \cong T_x M$$

image of $\nu_x^n M \cong T_x M$ under the diffeom. b/w the normal bdle and the tubular nbhd N_ε.

So essentially one needs to check that these two notions of orientation are dual to one another.

Now working locally:

$$\begin{tikzcd} u'' \arrow[d, mapsto] & H^n(M \times M) \arrow[r, "/[M]"] \arrow[d] & H^0 M \arrow[d] \\ 1 \times i_x^* u'' & H^n(\{x\} \times M) \arrow[r, "/[M]"] & H^0\{x\} \end{tikzcd}$$

$i_x : M \hookrightarrow M \times M$, $y \mapsto (x,y)$

$1 \times i_x^* u'' \longmapsto (1 \times i_x^* u'')/[M] = \langle i_x^* u'', [M] \rangle = 1$

<u>Formula for u'':</u>

<u>Theorem</u> (11.10 + 11.11)

If $b_1, \ldots, b_r \in H^*M$ is a basis (over $\mathbb{Q}$ or $\mathbb{Z}/2$), then $\exists$ a "dual" basis $b_1^\#, \ldots, b_r^\# \in H^*M$ such that

$$\langle b_i \cup b_j^\#, [M] \rangle = \delta_{ij} \quad \text{(Kronecker Delta)},$$

and then

$$u'' = \sum_{i=1}^{r} (-1)^{\overbrace{|b_i|}^{\text{dimension of coh. class}}} b_i \times b_i^\# .$$

<u>Pf:</u> We can write $u'' \in H^*(M \times M) \cong H^*M \otimes H^*M$ as $u'' = \sum_i a_i \times a_i'$ w/ $a_i, a_i' \in H^*M$. Now $a_i = \sum_j \lambda_{ij} b_j$ so $u'' = \sum_{i,j} \lambda_{ij} b_j \times a_i' = \sum_j b_j \times \underbrace{(\sum \lambda_{ij} a_i')}_{c_j}$, where $|b_j| + |c_j| = n$.

We checked that $(a \times 1) \cup u'' = (1 \times a) \cup u''$, so for any a w/ $|a| = |b_j| = n - |c_j|$ we have

$$((a \times 1) \cup u'')/[M] = ((1 \times a) \cup u'')/[M] = (1 \times a \cup (\sum b_j \times c_j))/[M]$$

$$\| \qquad\qquad = \sum_j \left((-1)^{|a||b_j|} b_j \times (a \cup c_j)\right)/[M]$$

$$a \cup \underbrace{u''/\mu}_{1} = a \qquad = \sum_j (-1)^{|a||b_j|} b_j \langle a \cup c_j, [M] \rangle$$

i.e. $\forall a \in H^*M$,

$$a = \sum_j (-1)^{|a||b_j|} b_j \langle a \cup c_j, [M] \rangle$$

letting $a = b_i$, we find

$$b_i = \sum_j (-1)^{|b_i||b_j|} b_j \langle b_i \cup c_j, [M] \rangle$$

and since $\{b_1, \ldots, b_r\}$ is a basis,

$$(-1)^{|b_i||b_j|} \langle b_i \cup c_j, [M] \rangle = \delta_{ij}.$$

We now set

$$\boxed{b_i^\# = (-1)^{|b_i|} c_i.}$$

(note $(-1)^{|b_i|} = (-1)^{|b_i||b_i|}$ b/c n is even $\Leftrightarrow n^2$ is even)

We have:

- $\langle b_i \cup b_i^\#, [M] \rangle = \langle b_i \cup (-1)^{|b_i|} c_i, [M] \rangle$

$$= (-1)^{|b_i|} \langle b_i \cup c_i, [M] \rangle$$

$$= (-1)^{|b_i||b_i|} \langle b_i \cup c_i, [M] \rangle = 1$$

- $u'' = \sum_i b_i \times c_i = \sum_i (-1)^{|b_i|} (b_i \times (-1)^{|b_i|} c_i)$

$$= \sum_i (-1)^{|b_i|} b_i \times b_i^\#.$$

$\square$

Finally:

<u>Theorem</u>: For any smooth, cpt, oriented mfld M^n, we have

$$\langle e(TM), [M] \rangle = \chi(M) \left(= \sum (-1)^i \operatorname{Rk} H^i M\right).$$

<u>Pf</u>: Recall that $e(TM) = \Delta^*(u'')$. Now

$u'' = \sum (-1)^{|b_i|} b_i \times b_i^\# \Longrightarrow e(TM) = \sum (-1)^{|b_i|} b_i \cup b_i^\#$

b/c

$$\begin{tikzcd} & p_1^* b_i \cup p_2^* b_i^\# \\ & b_i \times b_i^\# \arrow[u, phantom, "\shortparallel"] & \\ M \arrow[r, phantom, "b_i"] & M\times M \arrow[l, "p_1"'] \arrow[r, "p_2"] & M \arrow[l, phantom, "b_i^\#"] \\ & M \arrow[u, "\Delta"] \arrow[ul, equal] \arrow[ur, equal] & \end{tikzcd}$$

so $\Delta^*(b_i \times b_i^\#) = \Delta^* p_1^* b_i \cup \Delta^* p_2^* b_i^\#$
$= b_i \cup b_i^\#$.

Now $\langle e(TM), [M]\rangle = \langle \sum (-1)^{|b_i|} b_i \cup b_i^\#, [M]\rangle$

$$= \sum(-1)^{|b_i|}\langle b_i \cup b_i^\#, [M]\rangle$$

$$= \sum(-1)^{|b_i|} = \sum(-1)^k \cdot (\#\text{ of } b_i \text{ in } H^k M)$$

$$= \sum(-1)^k \operatorname{Rk} H^k M$$

$$= \chi(M). \qquad \square$$

Corollary: If M [oriented] admits a nowhere zero vector field, then $\chi(M)=0$

Pf: Multiplicativity of e: $e(TM) = e(\varepsilon^1)\cup e(\text{complement}) = 0$. $\square$

In fact, when M is not orientable, whole story goes through w/ $\mathbb{Z}/2$ coeffs, with $w_n(TM)$ in place of $e(TM)$. This requires Steenrod squares, though.

Next week:

Converse: If $e(TM)=0$, then M admits a nowhere-zero section.

"Euler class is the primary obstruction to existence of a nowhere-zero section over n-skeleton of the base."
(Here n = fiber dim'n.)