Lecture 22

The Thom Isomorphism Theorem

If $\begin{array}{c}\xi\\ \downarrow\\ B\end{array}$ is an oriented (real) n-plane bdle, then ∃ a unique class $u \in H^n(\xi, \xi_0; \mathbb{Z})$ which restricts to the orientation generator in $H^n(\xi_x, (\xi_x)_0; \mathbb{Z})$ for every $x \in B$. There is then an isom $H^k(B;\mathbb{Z}) \cong H^k(E,\mathbb{Z}) \xrightarrow[\cong]{\cup u} H^{n+k}(E,E_0;\mathbb{Z})$ for $k \geq 0$, and $H^*(E,E_0;\mathbb{Z}) = 0$ for $* < n$.

__Proof:__ We'll assume B is compact, or at least that there is a __finite__ open covering of B, $\{U_i\}$, s.t. $\xi|_{U_i} \overset{\text{or. preserving}}{\cong} U_i \times \mathbb{R}^n$. We have already observed that the Kunneth Thm implies that there is a unique class $u \in H^n(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0;\mathbb{Z})$ restricting to the orientation gen's, and moreover the Kunneth isom.

$$\begin{array}{ccccc} H^*(B;\mathbb{Z}) \otimes & H^n(\mathbb{R}^n,\mathbb{R}^n_0;\mathbb{Z}) & \xrightarrow{\cong} & H^{n+*}(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0;\mathbb{Z}) \\ \alpha & u & \longmapsto & \alpha\times u \end{array}$$

(which holds bc $H^*(\mathbb{R}^n,\mathbb{R}^n_0;\mathbb{Z}) \cong H^*(S^n;\mathbb{Z})$ is torsion-free) is just

$$H^k(B;\mathbb{Z}) \cong H^k(B\times\mathbb{R}^n;\mathbb{Z}) \xrightarrow{\cup u} H^{k+n}(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0;\mathbb{Z}).$$

This establishes the result when ξ is trivial.

Now say $B = \bigcup_{i=1}^{k} U_i$ and $\xi|_{U_i} \cong U_i\times\mathbb{R}^n$ for each i. By induction, we can assume the result holds for $\xi|_{\bigcup_{i=1}^{k-1} U_i}$.

It also holds for $\xi|_{U_k}$ and for $\xi|_{(\bigcup_1^{k-1} U_i) \cap U_k}$ b/c these last two bdles are trivial.

We now consider the relative Mayer-Vietoris sequence (letting $A' = \bigcup_1^{k-1} U_i$, $A = \bigcup_1^k U_k$)

$$\begin{aligned} H^{n-1}(\xi|_{A' \cap U_k}, (\xi_{A' \cap U_k})_0) \to H^n(\xi|_A, (\xi|_A)_0; \mathbb{Z}) \to H^n(\xi|_{A'}, (\xi|_{A'})_0; \mathbb{Z}) \oplus H^n(\xi|_{U_k}; (\xi|_{U_k})_0) \mathbb{Z}) \\ \longrightarrow H^n(\xi|_{A' \cap U_k}) (\xi|_{A' \cap U_k})_0; \mathbb{Z}) \to \end{aligned}$$

The map to the intersection term is the *difference* of the two restriction maps, so letting $u_{A'}$, u_k denote the Thom classes over $\xi|_{A'}$, $\xi|_{U_k}$, we have

$$u_{A'} \oplus u_k \longmapsto u_{A'}|_{A' \cap U_k} - u_k|_{A' \cap U_k}.$$

But since $u_{A'}$, u_k restrict to the orientation gen's on each fiber, the same is true of $u_{A'}|_{A' \cap U_k}$, $u_k|_{A' \cap U_k}$. Uniqueness of $u_{A' \cap U_k}$ then implies $u_{A'}|_{A' \cap U_k} = u_k|_{A' \cap U_k} = u_{A' \cap U_k}$. By exactness, $\exists\, u_A \in H^n(\xi|_A, (\xi|_A)_0; \mathbb{Z})$ mapping to $u_{A'} \oplus u_k$; it is unique b/c $H^{n-1}(\xi|_{A' \cap U_k}, (\xi|_{A' \cap U_k})_0) = 0$ (b/c this bdle is trivial, so Thm holds). Moreover, since $A' \cup U_k = A$, u_A restricts to the orientation gen at each point of A.

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It also follows from the LES that $H^*(\xi_A, (\xi_A)_0) = 0$ for $* < n$. To examine the map $\cup u$, we use a diagram of MV seq's:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k-1}(A'\cap U_k) \arrow[r, "\delta"] & H^k(A) \arrow[r] \arrow[d, "\cup u_A"] & H^k(A')\oplus H^k(U_k) \arrow[r] \arrow[d, "\cup u_{A'} \oplus \cup u_k"'] & H^k(A'\cap U_k) \arrow[r, "\delta"] \arrow[d, "\cup u_{A'\cap U_k}"] & H^{k+1}(A) \arrow[r] & \cdots \\
 & {} \arrow[r, "\delta"] & H^{n+k}(\xi|_A, (\xi|_A)_0) \arrow[r] & H^{n+k}(\xi|_{A'}, (\xi|_{A'})_0) \oplus H^k(\xi|_{U_k}, (\xi|_{U_k})_0) \arrow[r] & H^{n+k}(\xi|_{A'\cap U_k}, (\xi|_{A'\cap U_k})_0) \arrow[r, "\delta"] & {} &
\end{tikzcd}$$

The result now follows from the 5-lemma, although one does need to check that the square involving ∂ commutes. This is similar to the issue we encountered in the proof of the Proj Bdle Thm, and uses the formula:

$$\delta(y \cup i^*x) = \partial y \cup x \; ; \; i^*: H^*A'\cap U_k \to H^*A.$$

(See Lecture 11, where we proved this formula, and used it to check that a similar diagram commutes).

$\square$

Note: This thm goes through w/ $\mathbb{Z}/2$ coeff's for any bdle, replacing "or. gen" with "non-zero".

There are several other ways to think about the groups

$H^*(\xi, \xi_0)$: letting $D\xi = \{v\in\xi \mid |\xi| \leq 1\}$, $S\xi = \{v\in\xi \mid |v| = 1\}$

$\nwarrow$ "disk bdle" $\quad$ $\nwarrow$ "sphere bdle" $\quad$ $\swarrow$ refers to a chosen metric

and $T\xi = D\xi/S\xi$ (the "Thom Space" of ξ) we have

$$H^*(\xi, \xi_0) \underset{\nwarrow \text{excision + def. retraction}}{\cong} H^*(D\xi, S\xi) \underset{\nwarrow \{v \mid 1-\varepsilon < |v| \leq 1\} \text{ def. retracts to } S\xi.}{\cong} H^*(T\xi)$$

Corollary (The Gysin Sequence)

If $\begin{smallmatrix}\xi \\ \downarrow \pi \\ B\end{smallmatrix}$ is an oriented n-plane bdle over B, then there is a LES

$$\overset{\swarrow \text{Euler class}}{} \qquad \cdots \longrightarrow H^i B \xrightarrow{\cup e} H^{i+n} B \longrightarrow H^{i+n} \xi_0 \xrightarrow{\delta} H^{i+1} B \xrightarrow{\cup e} \cdots$$

(w/ $\mathbb{Z}$-coeffs).

Pf: This is essentially the LES of the pair (ξ, ξ_0):

$$\to H^{i+n}(\xi,\xi_0) \to H^{i+n}(\xi) \to H^{i+n}(\xi_0) \xrightarrow{\delta} H^{i+n+1}\xi \to$$

excepts that we've replaced $H^{i+n}(\xi,\xi_0) \to H^{i+n}\xi$ with

$$\overset{\swarrow \text{Thom class}}{}$$

$$\begin{tikzcd} H^i B \arrow[r, "\pi^*", "\cong"'] & H^i \xi \arrow[r, "\cup u", "\cong"'] \arrow[rr, bend right, "\cup u|_\xi"'] & H^{i+n}(\xi,\xi_0) \arrow[r, "|_\xi"] & H^{i+n}(\xi) \arrow[r, "(\pi^*)^{-1}", "\cong"'] & H^{i+n} B \end{tikzcd}.$$

This composite is really just $\cup e = \cup (\pi^*)^{-1}(u|_\xi)$ b/c:

$$\begin{tikzcd} H^*\xi \arrow[d, "\cup u|_\xi"'] & H^* B \arrow[l, "\pi^*"'] \arrow[d, "\cup (\pi^*)^{-1}(u|_\xi)"] \\ H^*\xi & H^* B \arrow[l, "\pi^*"] \end{tikzcd}$$

commutes: $\pi^*(\alpha \cup (\pi^*)^{-1} u|_\xi)$
$= \pi^*\alpha \cup u|_\xi$

So $(\pi^*)^{-1}(\pi^*\alpha \cup u|_\xi) = (\pi^*)^{-1}(\pi^*(\alpha \cup (\pi^*)^{-1} u|_\xi))$

$= \alpha \cup (\pi^*)^{-1} u|_\xi = \alpha \cup e$. $\square$

In this sequence, one often replaces ξ_0 by the sphere bdle $S\xi$, which ξ_0 deformation retracts to (again, we're assuming ξ has a metric). This LES is really the Serre spectral sequence for the fibration $\begin{smallmatrix} S\xi \\ \downarrow \\ B\end{smallmatrix}$...

To relate the Euler class to the Euler characteristic, we need to bring in some geometry.

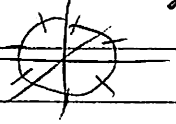
Theorem (MS 11.3)

Let $M^n \subset A$ be an embedded, closed submanifold of the Riemannian mfld A (that is, TA has a metric and $i: M \to A$ is a smooth homeomorphism onto its image). Then the map $H^k(A, A-M; \mathbb{Z}) \to H^k(A) \to H^k(M)$ sends a certain "Fundamental class" $u' \in H^k(A, A-M; \mathbb{Z})$ to the Euler class of the normal bdle $\nu^k M = \{v \in TA \mid v \perp TM\}$
$(k = \dim \nu^k)$

We'll need to describe this class u', which arises from the Thom class of the normal bdle of $M \subset A$. First, we need:

Tubular Nbhd Thm: If $M \subset A$ is a closed, embedded submfld of the Riemannian mfld A, then $\exists$ a nbhd $U \supset M$ which is diffeomorphic to the normal bdle $\nu^k(M) = \{v \in TA \mid v \perp TM\}$, and this diffeomorphism sends M to the zero section of $\nu^k M$.

Ex: $S^{n-1} \subset \mathbb{R}^n$:



U is a "spherical shell" around S^{n-1}.

We now have an excision isom

$$H^*(A, A-M; \mathbb{Z}) \cong H^*(\overset{\text{tubular nbhd}}{U}, U-M; \mathbb{Z}) \cong H^*(\nu^k M, (\nu^k M)_0; \mathbb{Z})$$

coming from excising the complement of U!! Note that the diffeomorphism $U \cong \nu^k M$ sends $M \to (\nu^k M)_0$.

We now <u>define</u> the fundamental class

$$u' \in H^*(A, A-M; \mathbb{Z})$$

to be the image of the Thom class of $\nu^k M$; here we must <u>assume</u> $\nu^k M$ is orientable.

We can now prove Thm 11.3:

<u>Pf of 11.3:</u>
We have

$$\begin{array}{ccccc} H^k(\nu^k, \nu^k_0) & \longrightarrow & H^k(\nu^k) & \xrightarrow{\overset{s = \text{zero section}}{s^*}} & H^k M \\ \underset{\text{Thom class}}{U} & \longmapsto & U|_{\nu^k} & \longmapsto & s^* U|_{\nu^k} = (\pi^*)^{-1} U|_{\nu^k} \overset{\text{def'n of Euler class}}{=} e_{\nu^k} \end{array}$$

(Note here that $\pi s = id_M$ so $s^* = (\pi^*)^{-1}$.)

So now the theorem follows immediately from the comm. diagram

$$\begin{tikzcd}
H^k(A, A-M) \arrow[r] \arrow[d, phantom, "\wr\!\!\parallel"] & H^k(A) \arrow[r] \arrow[d] & H^k(M) \arrow[d, equal] \\
H^k(U, U-M) \arrow[r] \arrow[d, phantom, "\wr\!\!\parallel"] & H^k(U) \arrow[r, "\cong"] \arrow[d, phantom, "\wr\!\!\parallel"] & H^k M \arrow[d, equal] \\
H^k(\nu^k, \nu^k_0) \arrow[r] & H^k(\nu^k) \arrow[r, "\cong"] & H^k(M)
\end{tikzcd}$$

$\square$

Application to Embeddings in $\mathbb{R}^N$:

If $M^n \hookrightarrow \mathbb{R}^N$ is an (closed) embedding with orientable normal bdle ν^k $(k = N-n)$ then we have shown that $e(\nu^k)$ is in the image of

(★) $H^k(\mathbb{R}^N, \mathbb{R}^N - M) \to H^k(\mathbb{R}^N) \to H^k M$.

But $H^k \mathbb{R}^N = 0$, so then $e(\nu^k M)$ must be zero as well.

Using Steenrod squaring op's, Milnor shows that the top Stiefel-Whitney class $w_k(\nu^k M)$ is in the image of this map (★) (w/ $\mathbb{Z}/2$ coeffs).

So then $w_k(\nu^k) = 0$. But we have $TM \oplus \nu^k M \cong T\mathbb{R}^N|_M = \varepsilon^N$,

so $w(TM) \cup w(\nu^k M) = w(\varepsilon^N) = 1$, i.e.

total Stiefel-Whitney class

$$w(\nu^k M) = w(TM)^{-1} \quad \text{(inverse in ring } H^*(M;\mathbb{Z}/2))$$

So one can solve for $w_i(\nu^k M)$ in terms of $w_i TM$, and we write $w_i(\nu^k M) = \bar{w}_i TM$ ("dual Stiefel-Whitney classes").

So if M^n embeds in $\mathbb{R}^{n+k}$, $\bar{w}_k TM = 0$ (this class depends only on M, not on the normal bdle).

Ex: $M = \mathbb{R}P^n$, $n = 2^r$. Then $\bar{w}_{n-1} T\mathbb{R}P^n = (w_1 \overset{\text{tautological bdle}}{\gamma_1^1})^{n-1} \neq 0$ $\mathbb{R}P^n$ doesn't embed in $\mathbb{R}^{2n-1}$. (We showed previously that in this case, $\mathbb{R}P^n$ doesn't immerse in $\mathbb{R}^{2n-2}$.) See Lecture 13 for the computation of $\bar{w}_{n-1}$.

Note that $\mathbb{R}P^2$ does immerse in $\mathbb{R}^3$, but (as we've just shown) doesn't embed.

So we've now related the Euler class to the normal bdle of an embedding; the next idea is:

Lemma 11.5: The normal bdle $\nu^n \left(\begin{array}{c} M \xhookrightarrow{\Delta} M\times M \\ m \longmapsto (m,m) \end{array} \right)$ is canonically diffeomorphic to the tangent bdle of M.

Proof: The map $D\Delta : TM \longrightarrow T(M\times M) \cong TM \times TM$ is just $v \longmapsto (v,v)$ (b/c the projections

$$\begin{tikzcd} & M \\ M \arrow[r, "\Delta"] \arrow[ur, bend left] \arrow[dr, bend right] & M\times M \arrow[u] \arrow[d] \\ & M \end{tikzcd}$$

are both id_M), so we just need to show that $\nu^n \cong \{(v,v) \in T(M\times M)\}$ (note that $D\Delta : TM \to T(M\times M)$ is a bdle map, hence induces an isom. onto its image).

A vector $(u,v) \in T(M\times M)$ is normal to ΔM

$\iff \langle (u,v), (w,w) \rangle = 0 \quad \forall\ w \in TM$

$\iff 0 = \langle u, w \rangle + \langle v, w \rangle = \langle u+v, w \rangle \quad \forall\ w \in TM$

$\iff u+v=0$.

So $\nu^n M$ is diffeomorphic to TM via

$$(v,-v) \longmapsto (v,v). \qquad \square$$

So now 11.3 says that

$$H^k(M\times M, M\times M - \Delta M) \to H^n(M\times M) \xrightarrow{\Delta^*} H^k(M)$$

sends the "fundamental class" u' to the Euler class of TM (assuming TM is orientable). The class $u'' = u'|_{M\times M}$ will be closely related to Poincaré Duality...