Lecture 21

Final Comments Re: K-theory

1) For any connected space X, we set $K^0(X) = [X, BU \times \mathbb{Z}]$

$K^1(X) = [X, \Omega(BU \times \mathbb{Z})]$, where $\Omega(BU\times\mathbb{Z}) = U = \operatorname{colim} U(n)$

Here we're observing that $\operatorname{colim} EU(n) \to \operatorname{colim} BU(n)$ is a universal $\operatorname{colim} U(n)$-bdle

so BU is the classifying space for U. Also note that for cpt X, $K^1(X) := \tilde{K}^0(SX) = [SX, BU] = [X, \Omega U]$ so this agrees.

2) The sequence of spaces

$$\mathbb{Z} \times BU, \ U, \ \mathbb{Z} \times BU, \ U, \ \cdots$$

is the "Bott Spectrum". This refers to the fact that we have weak eq's b/w the i^{th} space and the loop space of the $(i+1)^{st}$ space.

So "$\mathbb{Z} \times BU$ is an infinite loop space".

In any such situation one can construct a cohomology theory via Puppe sequences, i.e. given

$$k = \{X_0 \simeq \Omega X_1, \ X_1 \simeq \Omega X_2, \ X_2 \simeq \Omega X_3, \ \ldots\}$$

we $k_i(A) = \langle A, X_i \rangle \ (\cong \langle A, \Omega^2 X_{i+2} \rangle$, an abelian gp$)$

($\langle\ ,\ \rangle$ = basept preserving htpy classes of maps)

and then Puppe seq's yield LES in k_*.

The "coeff's" of the coh. theory are $k_*(pt) = \pi_0(X_*)$

So the coeff gps for K-theory are $\pi_0 \mathbb{Z}\times BU = \mathbb{Z}$, $\pi_0 U = 0$, $\mathbb{Z}, 0, \mathbb{Z}, 0, \cdots$

Oriented Bdles + the Euler Class (MS §9)

The Euler class is an *integral* char. class associated to *real* v. bdles: for any v. bdle $\mathbb{R}^n \to E \to X$, the Euler class is an element $e(E) \in H^n(X;\mathbb{Z})$, defined when E is "orientable".

Theorem (MS 11.12)
Let M^n be an orientable closed (smooth) mfld. Then the integer

$$\langle e(TM^n), [M] \rangle = \chi(M),$$

the Euler Characteristic of M (i.e. $\chi(M) = \sum_{i=0}^{n} (-1)^i \operatorname{Rk} H_i(M;\mathbb{Z})$, Rank).

(In particular, if M^n is orientable, then TM^n is an orientable **real** bdle)

Theorem: (MS 9.7)
If an orientable closed mfld (smooth) admits a nowhere vanishing tangent vector field then $\chi(M) = 0$.

Milnor proves this by showing that the Euler class is the "primary obstruction" to the existence of a nowhere zero section of $E \to X$ defined on the n-skeleton $X^{(n)}$! So if M^n has a nowhere-zero vector field, i.e. a nowhere-zero section of TM^n, then the primary obstruction must vanish, and then $\chi(M) = \langle e(TM^n), [M] \rangle = 0$.

<u>Example</u>: $\chi(S^{2n}) = 2 \neq 0$, so S^{2n} does not admit a nowhere vanishing vector field.

To begin, we need to discuss <u>orientations</u>.

<u>Def'n</u>: An <u>orientation</u> of a real v. space V is an equivalence class of ordered bases, under the equiv. rel'n

$$[v_1, \ldots, v_n] \sim [w_1, \ldots, w_n] \text{ if the transformation } v_i \mapsto w_i \text{ has determinant greater than zero.}$$

<u>Def'n</u> A vector bdle $\begin{matrix} \mathbb{R}^n \to E \\ \downarrow \\ X \end{matrix}$ is orientable if one of the following equivalent conditions is satisfied:

1) E admits a (continuous) orientation; that is, there exist orientations for each fiber and trivializations $E|_u \overset{\cong}{\to} U \times \mathbb{R}^n$ (over u) which send the chosen orientations to the standard orientation $[e_1, \ldots, e_n]$ of $\mathbb{R}^n$.

2) E admits transition fcns w/ positive det's (i.e. there is a trivialization in which <u>all</u> φ_{ij} have $\det \varphi_{ij} > 0$)

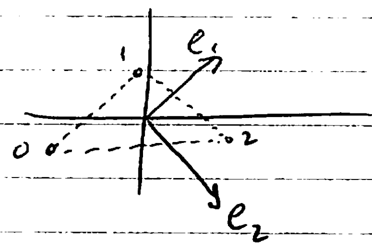
3) E is the mixed bdle associated to a principal $GL_n^+\mathbb{R}$-bdle, where $GL_n^+\mathbb{R} = \{A \in GL_n\mathbb{R} \mid \det A > 0\}$.

4) E is the mixed bdle associated to a principal $SO(n)$-bdle.

Pf (1) and 2) are equiv. essentially by defn; 2) and 3) are exactly the same by our prior discussion of mixed bdles / assoc. principal bdles, and 4) arises via Gram-Schmidt and various HW problems.)

Orientations on a real vector space V determine generators of the group $H_n(V, V_0; \mathbb{Z})$, where $V_0 = V - \{0\}$:
$H_n(D^n, S^{n-1}; \mathbb{Z}) \cong H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$

- Given an orientation $[e_1, -, e_n]$, choose a singular n-simplex $\Delta^n \to \mathbb{R}^n$ whose interior contains 0 and s.t. the vector pointing from its i^{th} vertex to its $(i+1)^{st}$ vertex is e_i:



Then this singular simplex lies in the relative cycles $Z_n(V, V_0)$, and maps to a generator of $H_n(S^n; \mathbb{Z})$.

There is then a dual generator of $H^n(V, V_0; \mathbb{Z}) \cong H^n(S^n; \mathbb{Z})$ determined [illegible] by the orientation.

More generally, if $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is an oriented v. bdle (i.e. we have <u>specified</u> an orientation on E) then <u>locally</u> we can choose consistent generators of $H_n(E_x, (E_x)_0; \mathbb{Z})$: locally $E|_U \cong U \times \mathbb{R}^n$, and (with the right choose of trivializations) the orientation of E_x maps to the standard orientation on $\mathbb{R}^n$. The cohomology gp $H^n(U \times \mathbb{R}^n, U \times \mathbb{R}^n_0; \mathbb{Z}) \cong H^n(U \times S^n; \mathbb{Z})$ now has a generator restricting to the orientation generator in each fiber, and we can transfer this generator back to $E|_U$.

Hence associated to the orientation, we have a covering $\{U_i\}$ of B and classes in $H^n(E|_{U}, E|_{U})_0; \mathbb{Z})$ which restrict in each fiber to the orientation generators. These classes can actually be glued together over all of E:

<u>Theorem</u> (The Thom Isom Theorem):

If E is an oriented real n-plane bdle, then $H^i(E, E_0; \mathbb{Z}) = 0$ for $i < n$, and $\exists$ a <u>unique</u> class $U \in H^n(E, E_0; \mathbb{Z})$ such that for every $x \in B$, the restriction

"Thom class"

$$u_x \in H^n(E_x, (E_x)_0; \mathbb{Z})$$

is the orientation generator. Moreover, the map

$$H^k(B;\mathbb{Z}) \cong H^k(E; \mathbb{Z}) \longrightarrow H^{n+k}(E, E_0; \mathbb{Z})$$
$$\alpha \longmapsto \alpha \cup u$$

↖ relative cup product

is an isomorphism.

(Note here that $H^k B \cong H^k E$ b/c $\overset{E}{\underset{B}{\downarrow}}{}^{\pi}$ is a fib'n with fiber $\mathbb{R}^n \simeq *$; hence π is a weak equivalence and induces isom's in H_* and H^*.)

There is now a restriction map $H^*(E, E_0) \to H^*(E, \emptyset) = H^* E$, giving rise to the <u>Euler class</u>:

<u>Def'n</u>: If $\overset{E}{\underset{B}{\downarrow}}{}^{\pi}$ is an oriented n-plane bdle, its Euler class is $e(E) = (\pi^*)^{-1}(U|_E) \in H^n(B; \mathbb{Z})$.

<u>Thm</u> (Property 9.5) The mod-2 reduction of the Euler class is precisely $w_n(E)$. That is, the map $H^n(B;\mathbb{Z}) \to H^n(B;\mathbb{Z}/2)$ sends $e(E)$ to $w_n(E)$.

With our def'n of the Stiefel-Whitney classes, this is not so clear, unfortunately. One can, however, define all the Stiefel-Whitney classes in terms of the Thom class, and then check that they satisfy the axioms (hence must agree w/ our def'n).

This requires the theory of "cohomology operations" and more specifically Steenrod squares.

<u>Basic Properties of the Euler Class:</u>

1) Naturality: If
$$\begin{array}{ccc} E & \to & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B' \end{array}$$
is an <u>orientation preserving</u> map of bdles, then $f^*(e(E')) = e(E)$.

(This follows from uniqueness of $u(E)$: in each fiber, the orientation generators agree under f^*, so $f^*(u(E'))$ has the defining property of $u(E)$.)

2) Changing the orientation causes $e(E)$ to simply change signs.

(Same reason as above)

3) $2 \cdot e(E) = 0$ if E is an odd-dim'l bdle.

(In odd dim'ns, $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto -x$ has negative det., hence is orientation reversing. Now for any bdle E,
$$\begin{array}{ccc} (E,-) & \xrightarrow{\cdot -1} & (E,+) \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$
$$-e_+(E) \underset{2)}{=} e_-(E) \underset{1)}{\longleftarrow\!\!\shortmid} e_+(E)$$
But the map is $B \xrightarrow{id} B$, so $-e_+(E) = e_+(E)$.)

Multiplicativity of the Euler Class:

Note that the Whitney sum formula says that if E is n-dim'l, F is m-dim'l, $w_{2n}(E \oplus F) = w_n E \cup w_m F$.

So after reducing mod 2, Euler classes are multiplicative. In fact:

Theorem (MS 9.6): $e(E_1 \oplus E_2) = e(E_1) \cup e(E_2)$, and $e(E_1 \times E_2) = e(E_1) \times e(E_2)$.

Pf: This is essentially a consequence of the Kunneth Theorem. One finds that the orientation generators are multiplicative, and the rest follows by naturality. □

(This can be used to show that the versions of Stiefel-Whitney classes defined via the Thom class satisfy the Whitney-Sum Formula.)

The Euler Class as an Obstruction

Thm: If $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is an orientable bdle w/ a nowhere-zero section, then $e(E) = 0$.

Pf: (Assume B is paracpt.) Choosing a metric on B, the span of the section and its orthogonal complement yield a splitting $E \cong E' \oplus \varepsilon'$. Then $e(E) = e(E') \cup e(\varepsilon')$, but $e(\varepsilon') = 0$ (b/c ε' can be pulled back from a point).