Bott Periodicity:

Finally we can complete the proof of Bott Periodicity and the Product Thm $K^0(S^2\times X)\cong K^0(S^2)\otimes K^0(X)$.

We have shown that for any bdle $[E,f]$ over $X\times S^2$, we have:

Laurent poly $\ell(x,z)=\sum_{n=-m}^{m} a_n(x)z^n$

$$[E,f]\cong[E,\ell]\cong[E,z^{-m}q]\cong[E,q]\otimes\pi_2^*H^{-m}$$

polynomial clutching fcn

in $K^0(X\times S^2)$

$$=\left([(n+1)E,\,L^n q]-[nE,\mathbb{1}]\right)\otimes\pi_2^*H^{-m}$$

linear clutching fcn

$$=\Big([((n+1)E)_+,\mathbb{1}]+\underbrace{[((n+1)E)_-,z]}_{[((n+1)E)_-,\mathbb{1}]\otimes\pi_2^*H}-[nE,\mathbb{1}]\Big)\otimes\pi_2^*H^{-m},$$

and this final expression lies in the image of

$$K^0(X)\otimes\mathbb{Z}[H]/(H-1)^2\longrightarrow K^0(X)\otimes K^0(S^2)\to K^0(X\times S^2).$$

So this map is surjective, as is the second map

$$K^0(X)\otimes K^0(S^2)\longrightarrow K(X\times S^2).$$

The first map is injective: if

$$\mathbb{Z}[H]/(H-1)^2\longrightarrow K^0(S^2)$$

is not injective, then some class $nH+m$ must be zero in $K^0(S^2)$. This means $n+m=0$, so we may write this class as $[nH]-[\varepsilon^m]$ for $n,m>0$. Now $[nH]-[\varepsilon^m]=0\iff nH\oplus\varepsilon^k\cong\varepsilon^{m+k}$ for some k. But this would imply $c(nH)=0$ (HW 2)

Which is impossible since $c_1(nH) = \overbrace{c_1(H) + \cdots + c_1(H)}^{n} \neq 0$.

<u>Corollary</u>: $K^0(S^2) \cong \mathbb{Z}[H]/(H-1)^2 \overset{\text{additively}}{\cong} \mathbb{Z} \oplus \mathbb{Z}$, and $\tilde{K}^0(S^2) \cong \mathbb{Z}$.

$$K^0(X) \otimes K^0(S^2) \xrightarrow{\mu} K^0(X \times S^2)$$

is always injective, then it will follow that

$$K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2 \xrightarrow{\cong} K^0(X) \otimes K^0(S^2) \xrightarrow{\cong} K^0(X \times S^2)$$

for any X (including $X = pt$).

<u>Claim</u>:
$$K^0(X) \otimes K^0(S^2) \xrightarrow{\mu} K^0(X \times S^2)$$

is 1) inj., 2) surj., 3) isom.

$$\iff \tilde{K}^0(X) \xrightarrow{\beta} \tilde{K}^0(S^2 X) \text{ is}$$

1) inj, 2) surj., 3) isom.

So in particular, since we've proven μ is surjective, β is surjective.

Assuming this claim, if we prove that β is injective, it will follow that μ is injective, hence an isomorphism, and then we'll get both the product Thm $K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2 \cong K^0(S^2 \times X)$ and the Bott Per. Thm $\tilde{K}^0(X) \underset{\beta}{\xrightarrow{\cong}} \tilde{K}^0(S^2 X)$. We'll prove that β is injective on finite CW cplxs via the Chern Character.

<u>Pf of Claim</u>: We need to re-examine our defn of $\beta: \tilde{K}^0(X) \longrightarrow \tilde{K}^0(S^2X)$:

$$\begin{tikzcd}
 & & \tilde{K}^0(S^2X) \arrow[d, hook] \\
\tilde{K}^0(X) \arrow[r, "\cong"] \arrow[urr, dashed, "\beta"] \arrow[drr, bend right] & \tilde{K}^0(S^2)\otimes\tilde{K}^0(X) \arrow[r, "\tilde{\mu}"] & \tilde{K}^0(S^2\times X) \arrow[d] \\
 & & \tilde{K}^0(S^2\vee X)
\end{tikzcd}$$

(is an b/c $\tilde{K}^0S^2 \cong \mathbb{Z}$)

$\alpha \longmapsto (H-1)\otimes\alpha$

$\boxed{\text{Composite is zero b/c it's zero in both } \tilde{K}^0S^2 \text{ and } \tilde{K}^0X}$

$\tilde{K}^0(S^2\vee X) \cong \tilde{K}^0(S^2)\oplus\tilde{K}^0(X)$

We showed that this is a split SES by examination of Puppe seq.

Since the 3-term seq. on RHS is exact, the horizontal map can be lifted uniquely to $\tilde{K}^0(S^2X)$, yielding the Bott map β.

We can now view β as a restriction of μ:

$$\begin{array}{ccccccc}
K^0(S^2)\otimes K^0(X) \cong (\tilde{K}^0S^2\oplus\mathbb{Z})\otimes(\tilde{K}^0(X)\oplus\mathbb{Z}) \cong & \tilde{K}^0S^2\otimes\tilde{K}^0X & \oplus & \tilde{K}^0X & \oplus & \tilde{K}^0S^2 & \oplus\,\mathbb{Z} \\
\downarrow\mu & \downarrow\tilde{\mu} & & \downarrow\cong & & \downarrow\cong & \downarrow\cong \\
K^0(S^2\times X) \cong \tilde{K}^0(S^2\times X)\oplus\mathbb{Z} \cong & \tilde{K}^0(S^2X) & \oplus & \underbrace{\tilde{K}^0X & \oplus & \tilde{K}^0S^2}_{\tilde{K}^0(X\vee S^2)} & \oplus\,\mathbb{Z}
\end{array}$$

(as indicated)

We claim that in this decomposition, the restriction of μ to the factor of $\tilde{K}^0X \subseteq K^0(S^2)\otimes K^0(X)$ maps isomorphically to the factor of $\tilde{K}^0X \subseteq K^0(S^2\times X)$, and the same for $\tilde{K}^0S^2$ and $\mathbb{Z}$. This will complete the proof of the claim.

The 3 arguments are similar, so let's focus on $\tilde{K}^0X$. The copy of $\tilde{K}^0(X)$ in $K^0(S^2)\otimes K^0X$ is really $\langle \varepsilon^1 - 0\rangle \otimes \tilde{K}^0(X)$, and

"subgp gen'd by"; K-theory class over S^2

$\mu((\varepsilon^n - 0)\otimes\alpha) = n(\pi_2^*\alpha) \in \tilde{K}^0(S^2\times X) \subseteq K^0(S^2\times X)$. In fact, this class is in the image of the splitting

$$\begin{array}{ccc} \tilde{K}^0(S^2\times X) & \longrightarrow & \tilde{K}^0(S^2\vee X) \\ & & \| \\ \pi_2^*\alpha & \longleftarrow & \tilde{K}^0(X)\oplus\tilde{K}^0(S^2). \\ & & \alpha \end{array}$$

So the restriction of μ to $\langle \varepsilon^1 - 0\rangle \otimes \tilde{K}^0(X)$ lands in the summand $\tilde{K}^0(X) \subseteq K(S^2\times X)$, and as a map $\mathbb{Z}\otimes\tilde{K}^0(X) \longrightarrow \tilde{K}^0(X)$, μ is just the isomorphism $n\otimes a \mapsto na$. $\square$

To complete the proofs of the Bott Per. Thm and the Product Formula, we need to establish injectivity of $\beta : \tilde{K}^0 X \longrightarrow \tilde{K}^0(S^2 X)$.

<u>Observation</u>: It suffices to prove injectivity for spheres (i.e. for $X = S^k$).

<u>Why?</u> If X is a finite complex, then $X = A \cup_\varphi D^k$ for some k. By induction on the # of cells, we can assume β is injective for A. Now the Puppe seq. for $A \subset X \to X/A \cong S^k$ is:

$$\begin{array}{ccccccc} \tilde{K}^0(S^2A) & \to & \tilde{K}^0(X/A) & \to & \tilde{K}^0(X) & \to & \tilde{K}^0(A) \\ \downarrow{\scriptstyle\beta_{S^2A}} & & \downarrow{\scriptstyle\beta_{X/A}} & & \downarrow{\scriptstyle\beta_X} & & \downarrow{\scriptstyle\beta_A} \\ \tilde{K}^0(S^4A) & \to & \tilde{K}^0(S^2(X/A)) & \to & \tilde{K}^0(S^2X) & \to & \tilde{K}^0(S^2A) \end{array}$$

This is exact, b/c it's just the continuation of the top Puppe sequence!

By induction, β_{S^2A}, $\beta_{X/A}$, β_A are injective (hence isomorphisms) [Note: S^2A is homotopy equiv. to a complex w/ same # of cells as A] so by the "4½ Lemma" β_X is injective too.

(Note that the diagram commutes b/c up to htpy, each square has the form
$$\begin{array}{ccc} \tilde{K}^0(Y) & \rightarrow & \tilde{K}^0(Z) \\ \downarrow \beta_Y & & \downarrow \beta_Z \\ \tilde{K}^0(S^2Y) & \rightarrow & \tilde{K}^0(S^2Z) \end{array}$$
for some $Z \subseteq Y$.)

$\square$

Finally, we have:

Prop'n: The Bott map $\beta: \tilde{K}^0 S^{2k} \to \tilde{K}^0 S^{2k+2}$ is an isom. for every k.

Pf: If $k=2n+1$ is odd, then we have surjections

$$\tilde{K}^0 S^1 \overset{\beta}{\twoheadrightarrow} \tilde{K}^0 S^3 \twoheadrightarrow \cdots \twoheadrightarrow \tilde{K}^0 S^{2n+1},$$

but $\tilde{K}^0(S^1) = 0$ (b/c all cplx bdles over S^1 are trivial). So for k odd, β is the zero map b/w trivial gps.

For $k=2n$, we use the Chern character:

$$\begin{array}{ccccccc} \tilde{K}^0 S^0 & \xrightarrow{\beta} & \tilde{K}^0 S^2 & \xrightarrow{\beta} & \tilde{K}^0(S^4) & \rightarrow & \cdots \\ \cong \downarrow ch & & \downarrow ch & & \downarrow & & \\ \tilde{H}^{even}(S^0;\mathbb{Z}) & \underset{\text{Kunneth}}{\overset{\cong}{\rightarrow}} & \tilde{H}^{even}(S^2;\mathbb{Z}) & \overset{\cong}{\rightarrow} & \tilde{H}^{even}(S^4;\mathbb{Z}) & \rightarrow & \end{array}$$

At each stage, we conclude
1) β is injective, hence an isom.
2) Ch is an isomorphism

[Initially, Ch maps to $\tilde{H}(\,;\mathbb{Q})$, but commutativity + induction shows it maps to $\tilde{H}(\,;\mathbb{Z})$]

$\square$

This completes the proof of Bott Per. + Prod Thm for X = finite CW cplx.

Here is the "universal" form of Bott Periodicity.

<u>Thm</u>: For cpt Hausdorff space X, there is an isom

$$\tilde{K}^0(X) \cong [X, BU] \quad (\text{and } K^0 X \cong [X, \mathbb{Z} \times BU])$$

where $BU = \operatorname{colim}_{n \to \infty} Gr_n \mathbb{C}^{2n}$ (under the maps $Gr_n \mathbb{C}^{2n} \to Gr_{n+1} \mathbb{C}^{2n+2}$ classifying $\gamma_n^{2n} \oplus \varepsilon_1$), and there is a weak equivalence $\Omega^2 BU \simeq \mathbb{Z} \times BU$. It follows that for any cpt Haus. X,

$$\tilde{K}^0(X) \cong \tilde{K}^0(S^2 X).$$

<u>Pf</u>: BU can be described instead as the larger colimit

$$\begin{array}{ccccccc}
Gr_1 \mathbb{C}^1 & \subseteq & Gr_1 \mathbb{C}^2 & \subseteq & Gr_1 \mathbb{C}^3 & \subseteq & Gr_1 \mathbb{C}^4 \ \cdots \\
 & & \cap\!| & & \cap\!| & & \cap\!| \\
 & & Gr_2 \mathbb{C}^2 & \subseteq & Gr_2 \mathbb{C}^3 & \subseteq & Gr_2 \mathbb{C}^4 \ \cdots \\
 & & & & \cap\!| & & \cap\!| \\
 & & & & Gr_3 \mathbb{C}^3 & \subseteq & Gr_3 \mathbb{C}^4 \ \cdots
\end{array}$$

Note: If X is disconnected, bdles can have varying fiber dim'ns. Such bdles are not classified by maps $X \to BU(n)$

or as the iterated colimit $\operatorname{colim}_n BU(n) = \operatorname{colim}_n (\operatorname{colim}_k Gr_n \mathbb{C}^{n+k})$.

When the base space X is ctd, bdles are all induced by maps $X \to BU(n)$ and the latter description easily gives $[X, BU] = \tilde{K}^0(X)$, bc maps $X \to BU$ factor through $BU(n)$ for some n (by cptness) and hence yield bdles well-defined up to stable isom. (Converse is the Bdle Htpy Thm).

Now Bott Periodicity says that for connected finite CW cplxs X (e.g. $Gr_n \mathbb{C}^{2n}$)

$[X, BU] \cong [S^2 X, BU]$. Since $\pi_1 Gr_n \mathbb{C}^\infty = \pi_1 BU(n) = \pi_0 U(n) = 0$ for all n,

based htpy classes of based maps

$\pi_1 \Omega^2 BU = \pi_3 BU = [S^3, BU] = [S^1, BU] = 0$

$$[X, BU] \cong [S^2 X, BU] = \langle S^2 X, BU \rangle = \langle X, \Omega^2 BU \rangle \cong [X, \Omega^2 BU]$$

Hence $\pi_k BU = \langle S^k, BU \rangle = [S^k, BU] = [S^k, \Omega^2 BU] = \pi_k \Omega^2 BU$, for $k > 0$, we need to exhibit a map $BU \to \Omega^2 BU$ inducing this isomorphism. This is easy: the inclusions $Gr_n \mathbb{C}^{2n} \hookrightarrow BU$ each correspond to a map $Gr_n \mathbb{C}^{2n} \to \Omega^2 BU$, well-defined up to htpy. At each stage, the diagram

$$\begin{array}{ccc} Gr_n \mathbb{C}^{2n} & \xrightarrow{f_n} & \Omega^2 BU \\ \cap\vert & \nearrow_{f_{n+1}} & \\ Gr_{n+1} \mathbb{C}^{2n+1} & & \end{array}$$

is homotopy commutative, but by the Htpy Ext'n property we can adjust f_{n+1} (up to htpy) to make the diagram commute. This yields a map

$$BU = \operatorname{colim}_n Gr_n \mathbb{C}^{2n} \longrightarrow \Omega^2 BU$$

which induces the above isom. in htpy. Since $\pi_0 BU = 0$ and $\pi_0 \Omega^2 BU = \pi_0 \Omega U = \pi_1 U = \operatorname{colim}_n \pi_1 U(n) = \mathbb{Z}$, we have a weak equivalence $\mathbb{Z} \times BU \to \Omega^2 BU$. [Note: all components of $\Omega^2 \mathbb{Z}$ are htpy equiv.]

Now, for any cpt Hausdorff space X, we get

$$\tilde{K}^0 X = [X, BU] \underset{\uparrow}{\cong} [X, \Omega^2 BU] \cong [S^2 X, BU] = \tilde{K}^0(S^2 X)$$

isom. b/c $BU \to \Omega^2 BU$
is actually a htpy equiv:
by Milnor's Thm, both have htpy
type of CW cplxs

<u>Conclude</u>: $\beta : \tilde{K}^0 X \to \tilde{K}^0 S^2 X$ is an isom. for any cpt Hausdorff space X.