Lecture 2

Basic Examples: Spheres and Projective Spaces

Def'n: $S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid \Sigma x_i^2 = 1\}$ (the unit sphere).

Fact: S^{n-1} is a smooth mfl'd.

Proof: We can cover S^{n-1} by local parametrizations arising from projection onto hyperplanes:

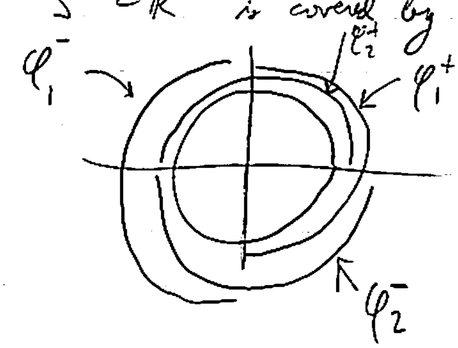
$$\varphi_i^{\pm} : D^{n-1} \xrightarrow{} S^{n-1}, \quad D^{n-1} = \{\vec{x} \in \mathbb{R}^{n-1} \mid \Sigma x_j^2 < 1\}$$

$$(x_1, \dots, x_{n-1}) \longmapsto (x_1, \dots, \underbrace{\pm\sqrt{1-\Sigma x_j^2}}_{i^{th}\text{ position}}, \dots, x_{n-1}).$$

This gives $2n$ local paramatrizations, since we can choose any $i \in \{1, \dots, n\}$ and either sign $+/-$.

These maps are homeomorphisms onto their images, because they are inverse to the relevant projections. The transition fcns $(\varphi_j^{\pm})^{-1}\varphi_i^{\pm}$ are then clearly smooth (since $\sqrt{y}$ is smooth away from $y = 0$). $\square$

Example: $S^1 \subset \mathbb{R}^2$ is covered by 4 charts:

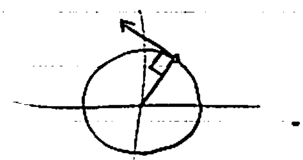


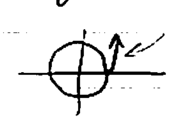
Prop'n: The tangent bundle to S^1 is trivial.

Proof: We have a continuous section $\begin{array}{c} S^1 \to T(S^1) \\ (x_1, x_2) \mapsto (x_2, -x_1) \end{array}$.

One can check (using MS Lemma 1.2) that $(x_2, -x_1) \in T_{(x_1, x_2)} S^1$. $\square$

Geometrically, this function just rotates (x_1, x_2) counter-clockwise by 90°:



This can be obtained more systematically from complex multiplication in $\mathbb{R}^2 \cong \mathbb{C}$: starting with the tangent vector $(0,1) = i$ in $T_{(1,0)} S^1$ ( i)

we simply multiply by $e^{i\theta} = (\cos\theta, \sin\theta)$ to obtain a tangent vector at $e^{i\theta} = (\cos\theta, \sin\theta) \in \mathbb{C} = \mathbb{R}^2$:

$$i e^{i\theta} = i(\cos\theta + i\sin\theta) = i\cos\theta - \sin\theta$$
$$= (-\sin\theta, \cos\theta).$$

So this process transports the vector $(0,1) \in T_{(1,0)} S^1$ to the vector $(-\sin\theta, \cos\theta) \in T_{(\cos\theta, \sin\theta)} S^1$. Letting $x_1 = \cos\theta$, $x_2 = \sin\theta$ gives the original formula.

This process really shows that any Lie group (= smooth mfld equipped with smooth mult'n + inverse maps) has trivial tangent bundle. (See MS p.20 for the case of S^3)

which is the group of unit quaternions).

Def'n (Real Projective Space)

$$\mathbb{R}P^n = S^n \Big/ {\scriptstyle \vec{x} \sim -\vec{x} \atop \text{for all } \vec{x} \in S^n} .$$

We equip $\mathbb{R}P^n$ with the quotient topology.

Fact: $\mathbb{R}P^n$ is a smooth mfld.

Proof: We can use the same charts as for S^n, except that we must compose with the projection $\pi : S^n \longrightarrow \mathbb{R}P^n$. Note that $\varphi_i^{\pm}(D^n)$ never contains two antipodal points, so $\pi \circ \varphi_i^{\pm} : D^n \longrightarrow \mathbb{R}P^n$ is still a homeomorphism onto its image. The transition maps are exactly the same as those for the sphere. $\square$

The canonical line bdle over $\mathbb{R}P^n$:

We have a line bdle γ_n^1 over $\mathbb{R}P^n$, whose fiber over $\{\vec{x}, -\vec{x}\} \in \mathbb{R}P^n$ consists of the line $\{c\vec{x} \mid c \in \mathbb{R}\} \subseteq \mathbb{R}^{n+1}$.

Formally, the total space of γ_n^1 is given by

$$E(\gamma_n^1) = \left\{ (\{\pm\vec{x}\}, \vec{v}) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \,\middle|\, \vec{v} = c\vec{x} \text{ for some } c \in \mathbb{R} \right\},$$

with the subspace topology. The projection is $\begin{array}{c} E(\gamma_n^1) \to \mathbb{R}P^n \\ (\{\pm x\}, v) \longmapsto \{\pm x\} \end{array}$

and each fiber has its natural vector space structure.
It is easy to check (MS p. 16) that γ_n^1 is
locally trivial (over the above coord nbhds for $\mathbb{R}P^n$, say).

<u>Theorem 2.1</u> γ_n^1 is not trivial, for any $n \geq 1$.

<u>Proof</u>: Trivial bundles $\begin{array}{c} V \\ \pi\downarrow \\ X \end{array}$ admit nowhere-zero sections $\begin{array}{c} V \\ s\uparrow\downarrow\pi \\ X \end{array}$ (with $\pi s = id_X$). If γ_n^1 had such a section s, then

$$\begin{array}{ccccc} S^n & \xrightarrow{\pi} & P^n & \xrightarrow{s} & E(\gamma_n^1) \\ x & \longmapsto & \{\pm x\} & \longmapsto & (\{\pm x\}, c(x)\cdot x) \end{array}$$

gives a continuous function $c: S^n \longrightarrow \mathbb{R}-\{0\}$ with $c(x) = -c(-x)$, but this contradicts the Intermediate Value Theorem. $\square$

<u>Rmk</u>: Continuity of c, and the fact that the obvious local trivializations of γ_n^1 are homeomorphisms, are essentially the same and can be proven just as in the proof of local triviality for tangent bdles.

Clutching Functions

Vector bundles can be built up from trivial bundles $U \times \mathbb{R}^n$ via clutching functions:

Def'n: If $V \xrightarrow{\pi} B$ is a vector bundle and $\varphi_1 : \pi^{-1}U_1 \xrightarrow{\cong} U_1 \times \mathbb{R}^n$, $\varphi_2 : \pi^{-1}U_2 \xrightarrow{\cong} U_2 \times \mathbb{R}^n$ are trivializations with $U_1 \cap U_2 \neq \emptyset$, then the isomorphism

$$\varphi_{21} := \varphi_2 \circ \varphi_1^{-1} : U_1 \cap U_2 \times \mathbb{R}^n \xrightarrow{\cong} U_1 \cap U_2 \times \mathbb{R}^n$$

is called a transition fcn for V. For each $x \in U_1 \cap U_2$, $\varphi_{21} : \{x\} \times \mathbb{R}^n \longrightarrow \{x\} \times \mathbb{R}^n$ is a linear isomorphism, so we may view φ_{21} as a mapping $U_1 \cap U_2 \to GL_n(\mathbb{R})$, called a clutching function.

Lemma (the cocycle condition):

If $U_1 \cap U_2 \cap U_3 \neq \emptyset$ and $\varphi_i : \pi^{-1}U_i \xrightarrow{\cong} U_i \times \mathbb{R}^n$, then we have

$$\boxed{\varphi_{32}\varphi_{21} = \varphi_{31}}$$

(either as comp. of fcns or as products of matrices)

Proof: $\varphi_{32} \circ \varphi_{21} = (\varphi_3 \varphi_2^{-1})(\varphi_2 \circ \varphi_1^{-1}) = \varphi_3 \circ \varphi_1^{-1} = \varphi_{31}$. □

We can now construct v. bdles using clutching fcns:

Prop'n: If $B = \bigcup_{i \in I} U_i$ (open sets) and $\varphi_{ji} : U_i \cap U_j \to GL_n \mathbb{R}$ are clutching fcns satisfying the cocycle condition, then

$$V = \left(\coprod_{i \in I} U_i \times \mathbb{R}^n \right) \Big/ (u, \vec{v}) \sim (u, \varphi_{ji} \vec{v}) \text{ for all } u \in U_i, i \in I$$

is a v. bdle over B w/ clutching fcns φ_{ji}.

Principal Bundles

When we have a vector bdle described in terms of clutching fcns, we can get rid of $\mathbb{R}^n$ entirely:

Say $\begin{matrix} V \\ \downarrow \\ B \end{matrix}$ has clutching fcns $\varphi_{ji} : U_i \cap U_j \to GL_n\mathbb{R}$. ($\{U_i\}$ an open cover of B)

Then we can use these fcns to construct a bundle whose fibers are $GL_n\mathbb{R}$ itself:

$$P = P_V = \Big(\coprod_i U_i \times GL_n\mathbb{R} \Big) \Big/ \sim$$

where $\underset{U_i \times GL_n\mathbb{R}}{\underset{\wedge}{(u, A)}} \sim \underset{U_j \times GL_n\mathbb{R}}{\underset{\wedge}{(u, \varphi_{ji}(u)A)}}$ if $u \in U_i \cap U_j$, $A \in GL_n\mathbb{R}$.

[We only glue pts from different elements of the disjoint union.]

The projections $U_i \times GL_n\mathbb{R} \to U_i \hookrightarrow B$ respect the equivalence reln, and yield a continuous projection map $P \xrightarrow{\pi} B$. Each fiber of this map is non-canonically homeomorphic to $GL_n\mathbb{R}$, and in fact $\pi^{-1}(U_i) \cong U_i \times GL_n\mathbb{R}$ for each i.

We can recover V by mixing: the space V admits a well-defined right action of $GL_n\mathbb{R}$, given by

$$(u, A) \cdot B = (u, AB).$$

<u>Lemma</u>: $P_V \times_{GL_n\mathbb{R}} \mathbb{R}^n \cong V$ as vector bdles over B.

Here $P_V \times_{GL_n\mathbb{R}} \mathbb{R}^n := (P_V \times \mathbb{R}^n) \Big/ \left(\begin{array}{l} (p,x) \sim (pA, A^{-1}x) \\ \text{for all } p \in P,\ x \in \mathbb{R}^n,\ A \in GL_n\mathbb{R} \end{array} \right)$.

<u>Proof</u>: We have a map

$$\begin{aligned} P \times \mathbb{R}^n &\longrightarrow V = \left(\coprod_i U_i \times \mathbb{R}^n\right)\Big/\sim\ ) \\ ([U_i, A], x) &\longmapsto [U_i, Ax] \end{aligned}$$

which is continuous and factors through the quotient on the left. On each fiber, it can be identified with the linear isomorphism $x \mapsto Ax$, and its inverse is given by the continuous map

$$\begin{aligned} P \times_{GL_n\mathbb{R}} \mathbb{R}^n &\longleftarrow \coprod_i U_i \times \mathbb{R}^n \\ [[U_i, I], x] &\longleftarrow (U_i, x)\ ) \end{aligned}$$

which factors through the equivalence rel'n on the right. $\square$

Note that the vector space structure on $(P \times_{GL_n\mathbb{R}} \mathbb{R}^n)|_b$ is just that inherited from $\mathbb{R}^n$.

These constructions allow us to pass between vector bundles and principal $GL_n\mathbb{R}$-bundles. To make the correspondence complete, we need a notion of <u>maps</u> b/w principal $GL_n\mathbb{R}$ bundles:

If $\begin{matrix} P_1 \\ \downarrow \pi_1 \\ B_1 \end{matrix}$ and $\begin{matrix} P_2 \\ \downarrow \pi_2 \\ B_2 \end{matrix}$ are principal $GL_n\mathbb{R}$-bundles,

then a <u>map</u> $P_1 \to P_2$ consists of a commutative diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{\tilde{\varphi}} & P_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ B_1 & \xrightarrow{\varphi} & B_2 \end{array}$$

in which $\tilde{\varphi}$ is <u>equivariant</u>: $GL_n\mathbb{R}$ acts on both P_1 and P_2, and $\tilde{\varphi}$ must satisfy

$$\tilde{\varphi}(p \cdot A) = \tilde{\varphi}(p) \cdot A$$

for all $p \in P_1$ and $A \in GL_n\mathbb{R}$.

<u>Exercise</u>: Maps b/w vector bdles induce maps b/w the associated $GL_n\mathbb{R}$-bundles, and vice-versa, and these correspondences respect composition of maps. Hence in particular, when applied to the identity map $V \xrightarrow{id} V$ we see that different trivializations produce the same $GL_n\mathbb{R}$ bundle up to isomorphism.

Metrics and Principal $O(n)$-bundles:

Recall that an inner product $\langle , \rangle$ on a real v.sp. V is a symmetric, bilinear function $V \times V \longrightarrow \mathbb{R}$, $v, w \longmapsto \langle v, w \rangle$ which is positive definite, i.e. $\langle v, v \rangle > 0$ for $v \neq 0$.

We want to consider vector bundles $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ equipped with a metric on each fiber.

Def'n: A Euclidean vector bundle is a real v. bdle $\begin{smallmatrix} E \\ \downarrow \pi \\ B \end{smallmatrix}$ together w/ a continuous function

$$\underbrace{E \times_B E}_{\{(v,w) \in E \times E \mid \pi(v) = \pi(w)\}} \xrightarrow{\langle , \rangle} \mathbb{R}$$

whose restriction to each subset $\pi^{-1}(b) \times \pi^{-1}(b) \subseteq E \times_B E$ $(b \in B)$ is an inner product on $\pi^{-1}(b)$.

Alternate Viewpoint: An inner product on a real v. space V is equivalent to a positive definite quadratic function $\mu : V \to \mathbb{R}$. This means $\mu(v) > 0$ for $v \neq 0$, and $\mu(v) = \sum l_i(v) l_i'(v)$ for some linear functions $l_i, l_i' : V \to \mathbb{R}$.

Given μ, we obtain an inner product by the "polarization" formula

$$\langle v, w \rangle = \tfrac{1}{2}\left(\mu(v+w) - \mu(v) - \mu(w)\right)$$

and given $\langle , \rangle$ we obtain the associated μ by setting

$$\mu(v) = \langle v, v \rangle =: |v|^2$$

Note that if we express v in an o.n. ("orthonormal") basis $\{e_i\}$, then

$$\mu(v) = \langle \Sigma \lambda_i e_i, \Sigma \lambda_i e_i \rangle = \sum_{i,j} \lambda_i \lambda_j,$$

and the functions $v \mapsto \lambda_i$ are all linear.

Now continuity of $\langle , \rangle : E \times_B E \to \mathbb{R}$ is equivalent to continuity of the associated $\mu : E \to \mathbb{R}$.

Lecture 3 A Euclidean bundle is not only locally isomorphic to a trivial bundle, it is also automatically <u>isometric</u> to a trivial bundle with its standard inner product. This is Lemma 2.4 in MS, and follows from continuity of the Gram–Schmidt orthogonalization process.

<u>Clutching Functions for Euclidean Bundles:</u>

<u>Proposition:</u> Let $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ be a Euclidean v. bdle. Then there exist local trivializations $\varphi_i : U_i \times \mathbb{R}^n \to E$ such that the associated clutching functions $\varphi_{ij} : U_i \cap U_j \to GL_n\mathbb{R}$ all land inside $O(n) = \{ A \in GL_n\mathbb{R} \mid AA^T = I_n \}$.