# Lecture 19 Bott Periodicity III: Linear Clutching Fcns

We have now shown that every bdle $[E,f]$ over $X\times S^2$ has the form $[E,\ell]$ for some Laurent poly. clutching fcn $\ell(x,z)=\sum_{-N}^{N} a_n(x)z^n$.

We now have
$$[E,\sum_{-N}^{N} a_n(x)z^n] = [E, z^{-N}(\sum_{n=-N} a_n(x)z^{n+N})]$$
$$= [E, \sum_{n=0}^{2N} a_n'(x)z^n]\otimes[\varepsilon', z^{-N}] = [E,q]\otimes \pi_2^* H^{-N}$$
$$a_n' = a_{n-N}$$

where $q=\sum_{n=0}^{2N} a_n'(x)z^n$ is a polynomial clutching fcn.

Prop. 2.6: If $q=\sum_{n=0}^{N} a_n(x)z^n$ is a poly clutching fcn, then $[E,q]\oplus[nE,\mathbb{1}]\cong[(n+1)E, L^n q]$ for some linear clutching fcn $L^n q = a_0' + a_1' z$.

PF: $[E,q]\oplus[nE,\mathbb{1}]\cong[(n+1)E, \begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & q\end{bmatrix}]$, and the matrices $\begin{bmatrix}1 & -z & & 0\\ & \ddots & \ddots & \\ 0 & & 1 & -z\\ a_n & \cdots & & a_0\end{bmatrix}$, $\begin{bmatrix}1 & & \\ & \ddots & \\ & & q\end{bmatrix}$ are equiv under row and column op's:

linear in $z$ as an endomorphism $(n+1)E\times S^1\to(n+1)E\times S^1$

$$\begin{bmatrix}1 & -z & & \\ & 1 & \ddots & \\ 0 & & 1 & -z\\ a_n & \cdots & & a_0\end{bmatrix} \overset{C_2\mapsto C_2+zC_1}{\sim} \begin{bmatrix}1 & 0 & & 0\\ 0 & 1 & -z & \\ & & \ddots & -z\\ a_n & za_n+a_{n-1} & \cdots & a_0\end{bmatrix} \overset{C_3\mapsto C_3+zC_2}{\sim} \cdots \begin{bmatrix}1 & & & \\ & 1 & & \\ & & \ddots & \\ a_n & za_n+a_{n-1} & \cdots & q\end{bmatrix} \sim \begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & q\end{bmatrix}$$

Each op'n is a htpy: for example, the first can be viewed as $C_2\mapsto C_2+tzC_1$ w/ $t=1$. So it remains to check that the first matrix is inv'ble. But in each fiber, choosing a basis we can view these row/col op's as "block" op'ns, i.e. $k$ op'ns if fiber dim'n is $k$, and each multiplies det by a non-zero scalar. So since $q$ is inv'ble, so is 1st matrix.

The proof of surjectivity of $K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \longrightarrow K(X \times S^2)$ will essentially be complete once we prove:

<u>Prop 2.7</u>: Given a bdle $[E, \underbrace{a(x)z + b(x)}_{\text{linear clutching fcn}}]$, there is a Whitney sum decomposition $E \cong E_+ \oplus E_-$ such that $[E, a(x)z + b(x)] \cong [E_+, \mathbb{1}] \oplus [E_-, z] \left( = \left[ E_+ \oplus E_-, \begin{bmatrix} \mathbb{1} & 0 \\ 0 & z \end{bmatrix} \right] \right)$.

<u>Proof</u>: First we eliminate the coefficient $a(x)$ via a htpy of clutching fcns: the homotopy

$$f_t = (1 + tz)\left[a(x)\frac{z+t}{1+tz} + b(x)\right]$$

has $f_0 = a(x)z + b(x)$, and is inv'ble for $t < 1$:

$(1 + tz) \neq 0$, and $\left|\frac{z+t}{1+t\bar{z}}\right| = \underbrace{|\bar{z}|}_{1}\left|\frac{z+t}{1+tz}\right| = \left|\frac{\bar{z}(z+t)}{1+tz}\right| = \left|\frac{1+t\bar{z}}{1+tz}\right| = 1.$

So for any $t \in [0,1)$, $[E, a(x)z + b(x)] \cong [E, f_t]$. The coeff. of $z$ in $f_t$ is $a(x) + tb(x)$, which is inv'ble for $t$ close to $1$: at $t = 1 \in S^1$ we know $a(x) + b(x)$ is a clutching fcn, and so $\det(a(x) + b(x))$ is bdd from below over all of $X$. The same is then true for $t \approx 1$. Now, we claim that

$$[E, a'(x)z + b'(x)] \cong [E, z + (a'(x))^{-1}b(x)]$$

if $a'(x)$ is inv'ble. More generally, if $g(x)$ is inv'ble, $[E, f(x,z)] \cong [E, g(x) f(x,z)]$.

This is easy to prove explicitly:

$$E\times D_0 \amalg E\times D_\infty \Big/ f \xrightarrow{Id \amalg g\times Id_{D_\infty}} E\times D_0 \amalg E\times D_\infty \Big/ gf.$$

This is well-defined b/c if $z\in S^1$,

$$\begin{array}{ccc} (e,z) & \longmapsto & (e,z) \\ \mathrel{\rotatebox{90}{$\equiv$}} & & \mathrel{\rotatebox{90}{$\equiv$}} \\ (f_z(e),z) & \longmapsto & (gf_z(e),z). \end{array}$$

$\boxed{\text{Note: It's very important here that } g(x) \text{ gives a well-defined isom } E\times D_\infty \xrightarrow{\cong} E\times D_\infty. \text{ If } g \text{ depended on } z\in D_\infty \text{ (e.g. } g = \text{mult'n by } z\text{) then } g \text{ might not be an isom at all pts in } D_\infty \text{ (e.g. at } z=0\text{).}}$

So we have now reduced to bdles of the form $[E, z+b(x)]$.

Note that $b(x): E\to E$ cannot have eigenvalues in $S^1$, or else $z-b(x)$ would become singular for those $z$.

Now we can decompose any fiber $E_x$ as follows:

$$E_x = (E_x)_+ \oplus (E_x)_-$$

sum of generalized $b(x)$-eigenspaces for eigenvalues $\lambda$ w/ $|\lambda|>1$

sum of generalized eigenspaces for $|\lambda|<1$

<u>Lemma 2.8</u>: The decompositions $E_x = (E_x)_+ \oplus (E_x)_-$ give rise to a Whitney sum decomposition

$$E = E_+ \oplus E_-$$

Assuming this, we complete the proof of 2.7.

On any fiber $E_x$, the map $b(x)$ preserves generalized eigenspaces (by def'n) so $b(x)$ sends $(E_x)_+ \to (E_x)_+$
$(E_x)_- \to (E_x)_-$.

Thus $b(x) = b_+(x) \oplus b_-(x)$ and

$$[E, z+b(x)] = \left[E_+ \oplus E_-, \begin{bmatrix} z+b_+(x) & 0 \\ 0 & z+b_-(x) \end{bmatrix}\right]$$

$$= [E_+, z+b_+(x)] \oplus [E_-, z+b_-(x)]$$

But $b_-(x)$ has all eigenvalues inside the unit disk, so $z + t\,b_-(x)$ is inv'ble for any $z \in S^1$, $t \in [0,1]$, and we get a htpy of clutching fcns $z + b_-(x) \simeq z$.

On the other hand, $b_+(x)$ has all eigenvalues <u>outside</u> the unit disk, so $zt + b_+(x)$ is always inv'ble.

$$\text{So } [E_+, z+b_+(x)] \cong [E_+, b_+(x)] \underset{\substack{\uparrow \\ \text{multiply by } (b_+(x))^{-1}}}{\cong} [E_+, \mathbb{1}]$$

and $[E_-, z+b_-(x)] \cong [E_-, z]$.

Thus we've shown $[E, a(x)z + b(x)] \cong [E_+, \mathbb{1}] \oplus [E_-, z]$ as desired. $\square$

<u>Proof of Lemma 2.8</u>: We may assume $E$ is trivial, so $b(x)$ is just a map $X \to M_n\mathbb{C}$. We need to show that the splittings $E_x = (E_x)_+ \oplus (E_x)_-$ vary continuously, in the sense that there are continuously varying bases for these subspaces, near some fixed $x \in X$.

Let $q(t)$ = char poly of $b(x)$. Then $q(t) = q_+(t)q_-(t)$ where $q_+, q_-$ are the char. poly's ~~restricted~~ of $b_+(x)$, $b_-(x)$. So $(E_x)_+ \subseteq \ker q_+(b(x))$ and $(E_x)_- \subseteq \ker(q_-(b(x)))$. Since $q_+, q_-$ are rel. prime in $\mathbb{C}[t]$, $q_+r + q_-s = 1$ for some $r, s \in \mathbb{C}[t]$, so $\ker q_+ \cap \ker q_- = 0$. Thus $(E_x)_+ = \ker q_+(b(x))$, $(E_x)_- = \ker q_-(b(x))$.

Now, we claim that $(E_x)_+ = \mathrm{Im}(q_-(b(x)))$, so that a basis for $(E_x)_+$ can be obtained from a linearly independent set in $E_x$ by applying $q_-(b(x))$ (and symmetrically for $(E_x)_-$). Since $q_+(b(x))\,q_-(b(x)) = 0$ (Cayley-Hamilton) we have $\mathrm{Im}\,q_-(b(x)) \subseteq \ker q_+(b(x))$, and since $rq_+ + q_-s = 1$, for $v \in \ker q_+(b(x))$ we have

$$v = rq_+v + q_-sv = q_-(sv)$$

so $v \in \mathrm{Im}\,q_-(b(x))$. Thus $(E_x)_+ = \ker q_+ = \mathrm{Im}\,q_-$, and similarly $(E_x)_- = \ker q_- = \mathrm{Im}\,q_+$.

Now we can choose a basis $v_1, \dots, v_n \in E_x = \mathbb{C}^n$ s.t.

$$\{q_-(b(x))v_1, \dots, q_-(b(x))v_k\},\ \{q_+(b(x))v_{k+1}, \dots, q_+(b(x))v_n\}$$

are bases for $(E_x)_+, (E_x)_-$. We claim that for $y$ near $x$,

$$\{q_-(b(y))v_1, \dots, q_-(b(y))v_k\},\ \{q_+(b(y))v_{k+1}, \dots, q_+(b(y))v_n\}$$

are bases for $(E_y)_+, (E_y)_-$. Assuming $q_-(by), q_+(by)$ are close to $q_-(bx), q_+(bx)$, these will still be linearly indep't sets (as will their union). Since

$$\{q_-(b(y))v_i\}_1^k \subseteq \operatorname{Im} q_-(b(y)) = \ker q_+(b(y)) = (E_y)_+$$
$$\{q_+(by)v_i\}_{k+1}^n \subseteq \operatorname{Im} q_+(by) = \ker q_-(by) = (E_y)_-$$

these sets are bases for $(E_y)_+, (E_y)_-$, so if we can show that the coeff's of the poly's $q_-(bx), q_+(bx)$ vary continuously w/ $x$, will be done.

The picture is:

$$\begin{array}{ccccc} M_n\mathbb{C} & \xrightarrow[\det(T-\lambda I)]{\text{coeff's of}} & \mathbb{C}^n & \xrightarrow{\text{roots}} & \operatorname{Sym}^n(\mathbb{C}) = \mathbb{C}\times\cdots\times\mathbb{C}/\Sigma_n \\ \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid \\ \left\{\begin{matrix}\text{Matrices w/}\\ \text{no eigenvalues}\\ \text{on } S^1\end{matrix}\right\} & \longrightarrow & W & \xrightarrow{\text{roots}} & \operatorname{Sym}^n(\mathbb{C}\setminus S^1) = \coprod_k \operatorname{Sym}^k D \times \operatorname{Sym}^{n-k}(\ldots) \\ & & & & \downarrow \\ & & & & \coprod_k \mathbb{C}^k \times \mathbb{C}^{n-k} \end{array}$$

$\mathbb{C}^n$ = Space of monic poly's of deg. $n$

$\Sigma_n$ = Symmetric gp

$W$ = the set of poly's w/o roots on $S^1$

sends $[\lambda_1, \dots, \lambda_k]$ to coeff's of $\prod(x-\lambda_i)$

Once we show that the map "roots" is continuous, it will follow that $q_+(bx), q_-(bx)$ vary continuously, b/c these are just the result of applying the lower composition in this diagram.
Continuity of "roots" is essentially Rouché's Thm in cplx analysis, or see Hatcher.