Bott Periodicity II: Reduction to Laurent Poly. Clutching Fcns

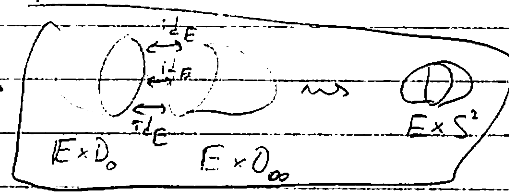
We are trying to show that the map

$$K(X)\otimes \mathbb{Z}[H]/(H-1)^2 \longrightarrow K(X)\otimes K(S^2) \longrightarrow K(X\times S^2)$$

$$E\otimes 1 \longmapsto E\otimes 1 \longmapsto \pi_1^* E$$

$$E\otimes H \longmapsto E\otimes H \longmapsto \pi_1^* E\otimes \pi_2^* H$$

is an isomorphism (when X is cpt Hausdorff). On the RHS, $\pi_1^* E = [E, \mathbb{1}]$



and $\pi_2^*(H) \cong [\varepsilon', z]$:

to see the second isom, recall that $H = [\varepsilon', z]$ as a bundle over S^2, and we have a diagram

$$\begin{array}{ccccc}
[\varepsilon', z]\to X\times S^2 & [x,z,w] & X\times D_0\times\mathbb{C} \amalg X\times D_\infty\times\mathbb{C} & \xrightarrow{(x,z,w)\sim(x,z,zw),\ z\in S^1} & X\times S^2 \\
& \downarrow & \downarrow & & \downarrow \\
[\varepsilon', z]\to S^2 & [z,w] & D_0\times\mathbb{C} \amalg D_\infty\times\mathbb{C} & \xrightarrow{(z,w)\sim(z,zw)} & S^2
\end{array}$$

for $z\in S^1$, $w\in\mathbb{C}$

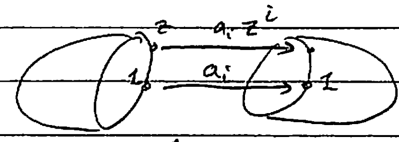
which is easily seen to be a pullback diagram.

Similarly, $\pi_2^* H^m = \pi_2^*[\varepsilon', z^m] = [\varepsilon', z^m]$ (where on the RHS $\varepsilon_1 = \begin{smallmatrix} X\times\mathbb{C} \\ \downarrow \\ X\end{smallmatrix}$), for any $m\in\mathbb{Z}$.

So to prove surjectivity, it will suffice to show that every bundle $E' = [E, F]$ over $X\times S^2$ has the form $\bigoplus_i [E_i, z^{m_i}]$ for some $m_i\in\mathbb{Z}$. (Note: it will then follow that the bdles $[E, z]$, $[E, \mathbb{1}]$ generate $K(X\times S^2)$ additively.)

<u>First Step:</u> replace $f: E\times S^1 \xrightarrow{\cong} E\times S^1$ by a <u>Laurent</u> poly clutching function $l=\sum_{i=-N}^{N} a_i z^i$,

i.e. $l(x,z)=\sum_{-N}^{N} a_i(x)\, z^i$



for some <u>endomorphisms</u>
$$\begin{tikzcd} E \arrow[rr, "a_i"] \arrow[dr] & & E \arrow[dl] \\ & X & \end{tikzcd}$$

$a_i(x): E_x \to E_x$ is <u>linear</u> for each x, but not necessarily an isomorphism

Clutching for $l = a_i z^i$.

<u>Prop'n 2.4:</u> For each
$$\begin{tikzcd} E\times S^1 \arrow[rr, "\cong"] \arrow[dr] & & E\times S^1 \arrow[dl] \\ & X\times S^1 & \end{tikzcd}$$
, $\exists\ l=\sum_{i=-N}^{N} a_i z^i$

s.t. $[E,f]\cong[E,l]$. Moreover, if l_0 and l_1 are Laurent poly'l clutching fcns that are h'pic through clutching fcns then they're homotopic through a <u>Laurent poly. clutching fcn homotopy</u> $l_t(x,z)=l(x,t,z)=\sum_{i=-N}^{N} a_i(x,t)\,z^i$

(w/ a_i an endom'n
$$\begin{tikzcd} E\times I \arrow[rr, "a_i"] \arrow[dr] & & E\times I \arrow[dl] \\ & X\times I & \end{tikzcd}$$
).

<u>Locally</u>, $f: E\times S^1 \to E\times S^1$ can be viewed as a family of linear isom's $S^1\times U\times\mathbb{C}^n \to S^1\times U\times\mathbb{C}^n$, i.e. $S^1\times U \to GL_n\mathbb{C}$.
We will show that fcns $S^1\times X \to GL_n\mathbb{C}$ can be approximated by Laurent poly's — more specifically by the partial sums of their Fourier series — and that if $l\approx f$ then

1) l is still a clutching fcn (i.e. bdle automorphism)
2) there is a "linear" homotopy $tl+(1-t)f$ from l to f which stays inside the space of clutching fcns.

These steps will follow from open-ness and local convexity of the space of clutching fcns, considered as a subset of the vector space $\mathrm{End}(E\times S^1)$

in the norm $\|\alpha\| = \sup_{|v|=1} |\alpha(v)|$ (where $|\cdot|$ denotes some metric on $E \times S^1$).

We now show that locally f can be approximated by Laurent poly's, starting with the case of a single fcn $X \times S^1 \to \mathbb{C}$

<u>Lemma 2.5</u>: For any $f: X \times S^1 \to \mathbb{C}$, let

$$a_n(X) = \frac{1}{2\pi}\int_0^{2\pi} f(X, e^{it}) e^{-int}\, dt$$

denote the Fourier coeff's of f. Then as $r \to 1$, the infinite sum

$$u(x,r,\theta) = \sum_{n \in \mathbb{Z}} a_n(x) r^{|n|} e^{in\theta}$$

converges uniformly (in x and θ) to f. In other words,

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $r \in (1-\delta, 1)$ then for all $(x,\theta) \in X \times [0, 2\pi]$

$$|f(x, e^{i\theta}) - u(x,r,\theta)| < \varepsilon.$$

Since the sum is convergent, it follows that $\forall \varepsilon > 0\ \exists \delta > 0, N \in \mathbb{N}$ s.t. $|f(x,e^{i\theta}) - \sum_{n=-N}^{N} a_n(x) r^{|n|} e^{in\theta}| < \varepsilon$ for all $r \in (1-\delta, 1)$ and all $(x,\theta) \in X \times [0,2\pi]$.

Proof: First we check that $\sum_{n \in \mathbb{Z}} a_n(x) r^{|n|} e^{in\theta}$ is convergent, so that u is well-defined. We have

$$|a_n(x)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(x,e^{it}) e^{-int}\, dt\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(x,e^{it})|\, dt \le C$$

where C is a bound on the cts fcn $f(x,z): X \times S^1 \to \mathbb{C}$ (note that X is compact, so a bd exists). Hence $|a_n(x) r^{|n|} e^{in\theta}| \le C r^{|n|}$, and $\sum a_n(x) r^{|n|} e^{in\theta}$ converges (for $r<1$) by comparison w/ $\sum r^{|n|}$.

Now,
$$u(x,r,\theta)=\sum_{n=-\infty}^{\infty}a_n(x)\,r^{|n|}e^{in\theta}=\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}f(x,e^{it})\,e^{-int}\,dt\right)r^{|n|}e^{in\theta}$$
$$=\sum_{n=-\infty}^{\infty}\int_0^{2\pi}\left(\frac{1}{2\pi}f(x,e^{it})\,|r|^n e^{in(\theta-t)}\,dt\right)$$
$$=\int_0^{2\pi}\left(\sum_{n=-\infty}^{\infty}\frac{1}{2\pi}f(x,e^{it})\,|r|^n e^{in(\theta-t)}\right)dt$$

b/c sum converges uniformly in t, by same argument as above.

$$=\int_0^{2\pi}f(x,e^{it})\underbrace{\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}r^{|n|}e^{in(\theta-t)}}_{P(r,\varphi)\ (\varphi=\theta-t)\ \text{("Poisson Kernel")}}\,dt$$
$$=\int_0^{2\pi}f(x,e^{it})\cdot P(r,\theta-t)\,dt.$$

Claim 1: $P(r,\varphi)=\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\varphi+r^2}$, so $P(r,\varphi)>0$ for all $0<r<1$, $\varphi\in\mathbb{R}$.

Pf: $2\pi\cdot P(r,\varphi)=\sum_{n=1}^{\infty}r^n e^{-in\varphi}+\sum_{n=0}^{\infty}r^n e^{in\varphi}$
$$=\frac{re^{-i\varphi}}{1-re^{-i\varphi}}+\frac{1}{1-re^{i\varphi}}=\frac{(1-re^{i\varphi})re^{-i\varphi}+(1-re^{-i\varphi})}{(1-re^{-i\varphi})(1-re^{i\varphi})}$$
$$=\frac{\cancel{re^{-i\varphi}}-r^2+1-\cancel{re^{-i\varphi}}}{1-\underbrace{re^{-i\varphi}-re^{i\varphi}}+r^2}=\frac{1-r^2}{1-r2\cos\varphi+r^2}.$$
$$e^{-i\varphi}+e^{i\varphi}=\cos(\varphi)\cancel{+i\sin(\varphi)}+\cos\varphi+i\sin\varphi$$
$$=2\cos\varphi$$

So $P(r,-)$ is 2π-periodic, even, monotone decreasing on $[0,\pi]$, and monotone increasing on $[\pi,2\pi]$.

Claim 2: $\int_a^{2\pi+a}P(r,\varphi)\,d\varphi=1$ for any $r<1$.

Pf: $\int_a^{2\pi+a}P(r,\varphi)\,d\varphi=\int_a^{2\pi+a}\left(\frac{1}{2\pi}\sum_{-\infty}^{\infty}r^{|n|}e^{in\varphi}\right)d\varphi=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}r^{|n|}\int_a^{2\pi+a}e^{in\varphi}\,d\varphi$
$$=\cancel{\frac{1}{2\pi}\sum_{n\neq 0}r^{|n|}\left.\frac{e^{in\varphi}}{in}\right|_a^{2\pi+a}}+\frac{1}{2\pi}\int_a^{2\pi+a}1\,d\varphi=1.\ \square$$

Now we examine the difference b/w F and u:

$$|u(x,r,\theta)-F(x,e^{i\theta})| = \Big|\int_0^{2\pi} P(r,\theta-t)\,F(x,e^{it})\,dt - F(x,e^{i\theta})\cdot\underbrace{\int_0^{2\pi} P(r,\theta-t)\,dt}_{1 \text{ by Claim 2}}\Big|$$

$$= \Big|\int_0^{2\pi} P(r,\theta-t)\,(F(x,e^{it})-F(x,e^{i\theta}))\,dt\Big|$$

$$\leq \int_0^{2\pi} P(r,\theta-t)\,|F(x,e^{it})-F(x,e^{i\theta})|\,dt.$$

Note $P>0$. We'll break this integral into 2 pieces.

Since $F: X\times S^1 \longrightarrow \mathbb{C}$ is continuous and X is cpt,

given $\varepsilon>0$, $\exists\ \delta_1>0$ $(\delta_1<\frac{\pi}{2})$ s.t. $|\theta-t|<\delta_1 \Rightarrow |F(x,e^{i\theta})-F(x,e^{it})|<\varepsilon$ for all $x\in X$.

Let $I_{\delta_1}=\int_{\theta-\delta_1}^{\theta+\delta_1} P(r,\theta-t)\,|F(x,e^{it})-F(x,e^{i\theta})|\,dt$, so that

$$0\leq I_{\delta_1} \leq \int_{\theta-\delta_1}^{\theta+\delta_1} P(r,\theta-t)\,\varepsilon\,dt \leq \left(\int_{\theta-\delta_1}^{\theta-\delta_1+2\pi} P(r,\theta-t)\,dt\right)\varepsilon=\varepsilon.$$

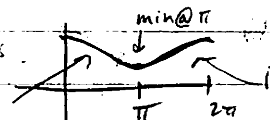
On the other hand,

$$I'_{\delta_1} \overset{\text{def'n}}{=} \int_{t=\theta+\delta_1}^{\theta-\delta_1+2\pi} P(r,\theta-t)\,|F(x,e^{it})-F(x,e^{i\theta})|\,dt = \int^{2\pi-\delta_1} P(r,\varphi)\cdot \overset{\substack{|F(x,z)|\leq C \\ \text{for } (x,z)\in X\times S^1}}{2C}\,d\varphi$$

$\varphi = t-\theta$, $P(r,\varphi)=P(r,-\varphi)$

$\leq 2C\cdot P(r,\delta_1)$.

graph of $P(r,-)$ is



increasing.

decreasing

symmetric

Now, as $r\to 1$, $P(r,\delta_1)=\dfrac{1-r^2}{1-2r\cos\delta_1+r^2} \longrightarrow \dfrac{0}{2-\underbrace{2\cos\delta_1}_{\cos\delta_1\neq 1}}=0$,

so $\exists\ \delta$ s.t. $r\in(1-\delta,1) \Rightarrow P(r,\delta_1)<\frac{\varepsilon}{2C}$.

Then $|u(x,r,\theta)-F(x,\theta)|\leq I_{\delta_1}+I'_{\delta_1} \leq \varepsilon+2C\frac{\varepsilon}{2C}=2\varepsilon$ for $r\in(1-\delta,1)$. $\square$

<u>Proof of Prop'n 2.4:</u> Choose a Hermitian metric $\langle , \rangle$ on E. Then $\mathrm{End}(E\times S^1)$ = set of bdle endomorphisms is a vector space under ptwise addition and scalar multn, and in fact it admits a norm: $\|\alpha\| = \sup_{|v|\leq 1} |\alpha(v)|$. $\leftarrow |\alpha(v)| = \sqrt{\langle \alpha v, \alpha v\rangle}$

We claim that

1) Balls in the $\|\cdot\|$ norm are convex: Say $t\in(0,1)$. If $\|\alpha-\gamma\|, \|\beta-\gamma\| < \varepsilon$, then $\|(t\alpha+(1-t)\beta)-\gamma\| < \varepsilon$;

2) $\mathrm{Aut}(E\times S^1) \subseteq \mathrm{End}(E\times S^1)$ is open.

Assuming 1) and 2), we now check that every bdle $[E,f]$ is isomorphic to $[E,\ell]$ for some Laurent poly. clutching fcn $\ell(x,z) = \sum_{n=-N}^{N} a_n(x) z^n$ with $a_n(x) \in \mathrm{End}(E)$.

Since $f \in \mathrm{Aut}(E\times S^1)$ and $\mathrm{Aut}(E\times S^1) \subset \mathrm{End}(E\times S^1)$ is open, $\exists\, \varepsilon>0$ s.t. $B_\varepsilon(f) \subseteq \mathrm{Aut}(E\times S^1)$. Moreover, if $g \in B_\varepsilon(f)$ is ~~any~~ other bdle automorphism, then for $t\in(0,1)$, $tf+(1-t)g \in B_\varepsilon(f)$ by 2), so $tf+(1-t)g$ is a htpy of clutching fcns connecting f to g, so $[E,f]\cong[E,g]$. So we will just need to show that Laurent poly's are dense in $\mathrm{End}(E\times S^1)$.

The proof of 1) is simple: $\|\alpha+\beta\| = \sup_{|v|=1} |\alpha(v)+\beta(v)| \leq \sup_{|v|=1} |\alpha v|+|\beta v|$

$\leq \sup_{|v|=1} |\alpha v| + \sup_{|v|=1} |\beta v| = \|\alpha\| + \|\beta\|$, i.e. $\|\cdot\|$ satisfies the Δ-ineq.

Now $\|(t\alpha+(1-t)\beta)-\gamma\| = \|(t\alpha-t\gamma)+(1-t)\beta-(1-t)\gamma\| \leq \|t(\alpha-\gamma)\| + \|(1-t)(\beta-\gamma)\| \leq t\|\alpha-\gamma\| + (1-t)\|\beta-\gamma\|$

$< \varepsilon t + (1-t)\varepsilon = \varepsilon$.

To prove 2) it suffices to show that

$$\inf_{(x,z)} \left|\det\left(\alpha_{(x,z)} : E_x \times \{z\} \to E_x \times \{z\}\right)\right| : End(E \times S^1) \to [0,\infty)$$

is continuous; then $Aut(E \times S^1)$ is the inverse image of $(0,\infty)$.

in the metric topology induced by $\|\cdot\|$

First note that $\det(\alpha_{(x,z)})$ is well-defined: it's independent of any chosen basis for $E_x \times \{z\}$ (b/c $\det(PAP^{-1}) = \det(A)$).

Now, say $\|\alpha - \beta\| < \delta$. Then if we choose an orthonormal basis (wrt $\langle , \rangle$) $\{e_i\}$ for $E_x \times \{z\}$, α and β are rep'd by matrices $A = (a_{ij})$ and $B = (b_{ij})$.

Since $(A-B)e_i = \Sigma(a_{ij} - b_{ij})e_j$ and $|e_i| = 1$, we have

$$\left|\Sigma(a_{ij} - b_{ij})e_j\right| \le \sup_{\substack{|v|=1 \\ v \in E_x \times \{z\}}} |(A-B)v| \le \sup_{\substack{|v|=1 \\ v \in E \times S^1}} |(\alpha-\beta)v| = \|\alpha - \beta\| < \delta$$

But $\langle e_i, e_j \rangle = \delta_{ij}$ so $|\Sigma(a_{ij} - b_{ij})e_j| = \Sigma|a_{ij} - b_{ij}|$.

Thus $|a_{ij} - b_{ij}| < \delta \ \forall i, j$, meaning the matrices A and B are element-wise close. Since $\det : M_n \mathbb{C} \to [0,\infty)$ is continuous, $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|a_{ij} - b_{ij}| < \delta \ \forall ij$ $\Rightarrow |\det A - \det B| < \varepsilon$.

Now if $\|\alpha - \beta\| < \delta$, then we have $\left||\det \alpha_{(x,z)}| - |\det \beta_{(x,z)}|\right| < \varepsilon$ for every $(x,z) \in X \times S^1$. Hence for some (x_0, z_0), (x_1, z_1)

$$\inf|\det \alpha| \ge |\det \alpha_{(x_0,z_0)}| - \varepsilon \ge |\det \beta_{(x_0,z_0)}| - 2\varepsilon \ge \inf|\det \beta| - 2\varepsilon$$

and $\inf|\det \beta| \ge |\det \beta_{(x_1,z_1)}| - \varepsilon \ge |\det \alpha_{(x_1,z_1)}| - 2\varepsilon \ge \inf|\det \alpha| - 2\varepsilon$

so $\|\alpha - \beta\| < \delta \Rightarrow \left|\inf|\det \alpha| - \inf|\det \beta|\right| \le 2\varepsilon$.

Finally we must apply the Lemma to show that Laurent poly's are dense in $(\mathrm{End}(E\times S^1), \|\cdot\|)$.

Choose a finite open covering $\{U_i\}$ of X such that there exist isometric trivializations $h_i : E|_{U_i} \xrightarrow{\cong} U_i \times \mathbb{C}^{n_i}$. [E may have different dim's over different comp's of X] Choose a part. of 1 $\{\varphi_i\}$ subordinate to $\{U_i\}$, with $X_i := \mathrm{supp}(\varphi_i)$ (a cpt set inside U_i).

Let $\alpha_i = \alpha|_{X_i \times S^1}$, and let $A_i = h_i \alpha_i h_i^{-1} : X_i \times S^1 \times \mathbb{C}^{n_i} \circlearrowleft$.

Then A_i is a map $X_i \times S^1 \to M_{n_i}\mathbb{C}$, and we can choose Laurent poly's $(l_i)_{k,l} : X_i \times S^1 \to \mathbb{C}$ which approximate the entries of A_i to within ε (uniformly over $X_i \times S^1$).

Let $l_i = ((l_i)_{k,l}) : X_i \times S^1 \times \mathbb{C}^{n_i} \circlearrowleft$, and by abuse of notation set $l_i = h_i^{-1} l_i h_i : E|_{X_i} \times S^1 \circlearrowleft$.

Then $\|\alpha - \sum \varphi_j l_j\| = \sup_{i,\,|v|=1} |\alpha v - \sum_j \varphi_j l_j v|$

Still a Laurent poly: $l_j = \sum_{-N_j}^{N_j} a_{j,n}(x) z^n \Rightarrow \sum \varphi_j l_j = \sum (\varphi_j a_{j,n}) z^n$

$= \sup_{i,\,|v|=1} |\sum_j \varphi_j \alpha v - \sum_j \varphi_j l_j v| \leq \sup_{i,\,|v|=1} \sum_{j : \varphi_j(x) \neq 0} |\alpha v - l_j v|$

$(\sum \varphi_j \equiv 1)$

But if $v = \sum v_k e_k$ ($\{e_k\}$ an o.n. basis for fiber containing v) then $|v_k| \leq 1$ for each k and $|(\alpha - l_j)(\sum v_k e_k)| \leq \sum |(\alpha - l_j) e_k|$

$= \sum |((A_j)_{ik} - (l_j)_{ik}) e_i| \leq n_i^2 \cdot \varepsilon$.

$(i,k)^{th}$ entries

So $\|\alpha - \sum \varphi_j l_j\| \leq n_i^2 \cdot \varepsilon$. So for ε small, $\sum \varphi_j l_j$ is invertible and homotopic to α. For X = CW cplx, we won't need the part about Laurent homotopies. □