Lecture 17: Bott Periodicity (Part I)

We'll follow Hatcher Chapter 2 rather closely, and these notes may be less complete than in previous lectures.

The Fundamental Product Theorem:

We'll deduce that the Bott Periodicity map

$$\beta : \tilde{K}^0(X) \longrightarrow \tilde{K}^0(S^2X)$$

is an isom (for X cpt Hausdorff) by studying the K-theory of $X \times S^2$. We'll prove:

__Thm:__ The external tensor product gives an isomorphism

$$K^0(X) \otimes K^0(S^2) \longrightarrow K^0(X \times S^2).$$
$$x \otimes y \longmapsto \pi_1^* x \otimes \pi_2^* y$$

To do this we'll study bundles over $X \times S^2$ by decomposing $S^2 = \mathbb{C} \cup \{\infty\}$ into $D_0 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $D_\infty = \{z \in \mathbb{C} \mid |z| \geq 1\} \cup \{\infty\}$. The homotopy equivalences

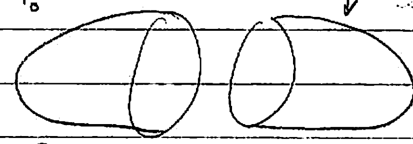
$$X \times D_0 \simeq X \simeq X \times D_\infty$$

imply that any bundle $\begin{matrix} E' \\ \downarrow \\ X \times S^2 \end{matrix}$ restricts to a "product bundle" on both $X \times D_0$ and $X \times D_\infty$; that is if $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ is $\begin{matrix} E'|_{X \times \{1\}} \\ \downarrow \\ X \end{matrix}$, then

$E_0 = E'|_{X \times D_0} \cong E \times D_0 \cong E'|_{X \times D_\infty} \equiv E_\infty$

Moreover, we can reconstruct E' by clutching:

$$E\times D_0 \xleftarrow[h_0]{\cong} E_0 \qquad E_\infty \xrightarrow[h_\infty]{\cong} E\times D_\infty$$



$D_0\times X$ $\qquad$ $D_\infty \times X$

Now we have homeomorphisms

$$E' \cong \frac{E\times D_0 \amalg E\times D_\infty}{\substack{(e,z)\sim h_\infty h_0^{-1}(e,z) \\ \text{for } z\in D_0\cap D_\infty = S^1}} \cong \frac{E\times \tilde{D}_0 \amalg E\times \tilde{D}_\infty}{\substack{(e,z)\sim \tilde{h}_\infty \tilde{h}_0^{-1}(e,z) \\ \text{for } z\in \tilde{D}_0\cap \tilde{D}_\infty}}$$

where on the right, $\tilde{D}_0$ and $\tilde{D}_\infty$ are open neighborhoods of D_0, D_∞ (and $\tilde{h}_0, \tilde{h}_\infty$ are extensions of h_0, h_∞ to these nbhds). It's easy to check that the RH map is a homeomorphism and hence the middle space is a v.bdle. So now we can conclude all the spaces are the same.

Note: in this construction, we can always assume that $h_0 : E \to E$ ($E = E'|_{X\times\{1\}}$) and $h_\infty : E \to E$ are the identity: if h_0 isn't to begin with, then

$$E_0 \xrightarrow{h_0} E\times D_0 \xrightarrow[\cong]{(h_0|_E)^{-1}\times id} E\times D_0$$

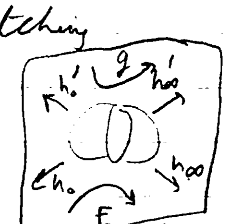
gives a replacement which is.

Our basic approach will be to replace arbitrary clutching fcns by simpler and simpler ones which yield the same bdle $\begin{smallmatrix} E' \\ \downarrow \\ X\times S^2 \end{smallmatrix}$. The basic tool is:

__Lemma:__ If $f: E\times S^1 \to E\times S^1$ is any any bdle isom (over $X\times S^1$) then $[E,f] := E\times D_0 \amalg E\times D_\infty \big/ (e,z)\sim f(e,z)$ is a vector bdle, and if f and f' are homotopic through bdle isom's, then the resulting bdles E' and E'' are isomorphic.
Moreover, if $[E,f]\cong[E,g]$, then $f\simeq g$ through clutching fcns, which may be taken to be the identity over $1\in S^1$ if f, g are.

__Proof:__ $E' \cong E\times\tilde{D}_0 \amalg E\times\tilde{D}_\infty \big/ (e,z)\sim\left(\pi f\left(e,\frac{z}{|z|}\right), z\right)$

where $\tilde{D}_0, \tilde{D}_\infty$ are open nbhds of D_0, D_∞; the RHS is clearly a v. bdle. Moreover, if f_t is a htpy of clutching fcns, $\xi = E\times D_0\times I \amalg E\times D_\infty\times I \big/ (e,z,t)\sim(f_t(e,z),t)$



is a bdle htpy connecting the two clutched bdles for f_0 and f_1.
If $[E,f]\cong[E,g]$ then $(E\times D_0 \amalg E\times D_\infty)/f$ has trivializations $h_0, h_0', h_\infty, h_\infty'$ w/ $h_\infty h_0^{-1} = f$, $h_\infty'(h_0')^{-1} = g$ and $h_0 h_0'^{-1}$, $h_\infty h_\infty'^{-1}$ are nullhtpic as maps $D_0, D_\infty \to \mathrm{Aut}\,E$. So $g = h_\infty' h_\infty^{-1} h_\infty h_0^{-1} h_0 (h_0')^{-1} \simeq h_\infty h_0^{-1}$

__Example:__ consider the case $X = pt$. Then we want to describe the tautological bdle $\begin{smallmatrix} H=\gamma_1^1 \\ \downarrow \\ \mathbb{C}P^1 = S^2 \end{smallmatrix}$ via clutching.

We need to describe trivializations of $\begin{smallmatrix} H|_{D_0} \\ \downarrow \\ D_0 \end{smallmatrix}$, $\begin{smallmatrix} H|_{D_\infty} \\ \downarrow \\ D_\infty \end{smallmatrix}$

We have $D_0 = \{ [u,w] \in \mathbb{CP}^1 \mid |\frac{u}{w}| \leq 1 \}$

$$D_\infty = \{ [u,w] \in \mathbb{CP}^1 \mid |\frac{u}{w}| \geq 1 \}$$

and we have sections

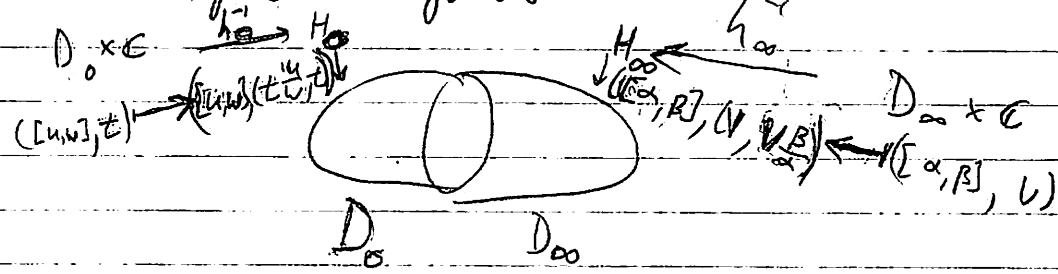
$$D_0 \longrightarrow H|_{D_0}$$
$$[u,w] \longmapsto \left([u,w], \left(\frac{u}{w}, 1\right)\right) \quad \text{(continuous, since } w \neq 0\text{)}$$

$$D_\infty \longrightarrow H|_{D_\infty}$$
$$[u,w] \longmapsto \left([u,w], \left(1, \frac{w}{u}\right)\right) \quad \text{(continuous, since } u \neq 0\text{)}.$$

These sections are clearly nowhere-zero, and hence trivialize $H|_{D_0}$ and $H|_{D_\infty}$. The clutching fcn

$$f : \mathbb{C} \times S^1 \longrightarrow \mathbb{C} \times S^1$$

is then computed as follows:



So the composite $h_\infty h_0^{-1}$ sends

$$([u,w], t) \longmapsto \left([u,w], \left(t\frac{u}{w}, t\right)\right) \longmapsto \left([u,w], t\frac{u}{w}\right)$$

$$|\tfrac{u}{w}| = 1$$

Letting $z = \frac{u}{w}$, this clutching fcn is $S^1 \times \mathbb{C} \longrightarrow S^1 \times \mathbb{C}$, $(z, t) \longmapsto (z, zt)$, which we write simply as $f(z) = z$ as a map $S^1 \to GL_1\mathbb{C}$.

Note: Hatcher seems to have this backwards. He doesn't exactly explain his conventions on clutching fcns on p. 23, but there is at least an inconsistency between his choice on p. 23 and his choice on p. 43 (where he writes $f = h_0 h_\infty^{-1}$

Arithmetic of Clutching Functions

1) If $f: E_1 \times S^1 \to E_1 \times S^1$, $g: E_2 \times S^1 \to E_2 \times S^1$ are clutching fcns, then $f \oplus g : E_1 \oplus E_2 \times S^1 \to E_1 \oplus E_2 \times S^1$ is a clutching fcn, and $[E_1, f] \oplus [E_2, g] \cong [E_1 \oplus E_2, f \oplus g]$ as bdles over $X \times S^2$.

2) If $f: E \times S^1 \to E \times S^1$ is a clutching fcn, and $g: S^1 \to GL_1 \mathbb{C} = \mathbb{C}^*$ is any map, then

$$fg : E \times S^1 \longrightarrow E \times S^1$$
$$(e, z) \longmapsto (\pi_1 f(e,z) \cdot g(z), z)$$

is a clutching fcn, and $[E, fg] \cong [E, f] \otimes [\varepsilon^1, g]$ where $[\varepsilon^1, g]$ is the line bdle over $X \times S^2$ clutched from the trivial line bdle over $X \times S^1$ via g.

<u>Key Computation</u>: (Ex. 1.13)

$$H \otimes H \oplus 1 \cong H \oplus H.$$

<u>Pf</u>: By 2), $H \otimes H = [\varepsilon^1, z] \otimes [\varepsilon^1, z] = [\varepsilon^1, z^2]$

and by 1), $[\varepsilon^1, z^2] \oplus \varepsilon^1 = [\varepsilon^1, z^2] \oplus [\varepsilon^1, 1] \cong [\varepsilon^1 \oplus \varepsilon^1, z^2 \oplus 1]$

In other words, $H \otimes H \oplus 1$ is formed by clutching a trivial 2-plane bdle via the matrix $\begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix}$.

But we have a htpy

through $P_t \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} P_t^{-1} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$ where P_t is a path from I to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix},$$

so by the lemma

$$H \otimes H \oplus 1 \cong [\varepsilon^2, z^2 \oplus 1] \cong [\varepsilon^2, z \oplus z] \cong [\varepsilon^1, z] \oplus [\varepsilon^1, z] = H \oplus H.$$

We'll actually show that this equation completely determines $K^0(S^2)$; that is

$$K^0(S^2) \cong \mathbb{Z}[H] / (H^2 - 2H + 1) = \mathbb{Z}[H] / (H-1)^2.$$

This will actually be an important part of our proof of the product Theorem $K^0(S^2) \otimes K^0(X) \cong K^0(X \times S^2)$:

we'll show that $K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K^0(X) \otimes K^0(S^2) \to K^0(X \times S^2)$ is an isom for any X.

Laurent Polynomial Clutching Functions:

We call a clutching function $l : E\times S^1 \to E\times S^1$ a L.P.C.F. if
$$l(x,z) = \left(\sum_{i=-n}^{n} a_i(x)\, z^i,\ z\right)$$
for some (fiber-wise linear) endomorphisms $a_i : E \to E$.

Here $a_i(x)\cdot z^i : E_x \to E_x$ is just $a_i(x) : E_x \to E_x$ multiplied by $z \in \mathbb{C}$.

We will show that any bdle $\begin{matrix} E' \\ \downarrow \\ X\times S^2 \end{matrix}$, there is some Laurent Poly. l s.t. $E' \cong [E, l]$.

<u>Basic Idea</u>: $E' \cong [E, f]$ for some f, and we can approximate f by some partial sum of its "Fourier Series". Then we will connect f to this Laurent poly. by a linear htpy.

We can then reduce any Laurent poly. to a linear function via ~~stabilization~~ (Hatcher 2.6)