# Lecture 16

We're trying to show that

$$Ch: K^0X\otimes\mathbb{Q} \to H^{even}(X;\mathbb{Q})$$

is an isomorphism whenever $X$ is a finite CW cplx.

We'll start with the case of spheres:

**Prop'n**: $ch: K^0S^{2n}\otimes\mathbb{Q} \to H^{even}(S^{2n};\mathbb{Q})$ is an isomorphism.

Proof: We'll show that the reduced Chern character

$$(1) \qquad ch: \tilde{K}^0(S^{2n}) \to \tilde{H}^{even}(S^{2n};\mathbb{Q}) \quad (\cong H^{2n}(S^{2n};\mathbb{Q}) \text{ if } n>0)$$

in injective, with image $\tilde{H}^{even}(S^{2n};\mathbb{Z}) \subseteq \tilde{H}^{even}(S^{2n};\mathbb{Q})$. Then up to isomorphism, (1) is just the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$, and after tensoring the LHS with $\mathbb{Q}$ we get an isomorphism. Since ch is clearly an isomorphism when the space in question is a point, the diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{K}^0(S^{2n})\otimes\mathbb{Q} \arrow[r] \arrow[d, "\cong\ ch"'] & K^0S^{2n}\otimes\mathbb{Q} \arrow[r] \arrow[d, "ch"] & K^0(pt)\otimes\mathbb{Q} \arrow[r] \arrow[d, "ch\ \cong"] & 0 \\
0 \arrow[r] & \tilde{H}^{even}(S^{2n};\mathbb{Q}) \arrow[r] & H^{even}(S^{2n};\mathbb{Q}) \arrow[r] & H^0(pt;\mathbb{Q}) \arrow[r] & 0
\end{tikzcd}$$

completes the proof. (Note that the top row remains exact after tensoring with $\mathbb{Q}$ b/c $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.)

We'll now study (1) by induction on $n$, using Bott periodicity. For $n=0$, (1) is clearly just $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

Assuming the result for $S^{2n}$, we have the diagram

$$(2)\qquad \begin{array}{ccc} \tilde{K}^0(S^{2n}) & \xrightarrow[\cong]{\beta} & \tilde{K}^0(S^2(S^{2n})) \\ \big\downarrow ch & & \big\downarrow ch \\ \tilde{H}^{even}(S^{2n};\mathbb{Q}) & \xrightarrow[\cong]{\sigma} & \tilde{H}^{even}(S^2(S^{2n});\mathbb{Q}) \end{array}$$

note that $S^2(S^{2n}) = S^{2n+2}$

Where the bottom map arises from the cross product

$$\tilde{H}^2(S^2;\mathbb{Q}) \otimes H^{2n}(S^{2n};\mathbb{Q}) \xrightarrow{\cong} H^{2n+2}(S^2 \times S^{2n};\mathbb{Q})$$

isom by Kunneth thm

in exactly the same way that $\beta$ arises from tensor product.

To be precise, the long exact sequence for $S^2 \vee S^{2n} \subseteq S^2 \times S^{2n}$

gives

$$\begin{array}{ccccc} H^2S^2 \otimes H^{2n}S^{2n} & \longrightarrow & H^{2n+2}(S^2\times S^{2n}) & \longrightarrow & \underbrace{H^{2n+2}(S^2 \vee S^{2n})}_{0} \\ \alpha \otimes \beta & \longmapsto & \pi_1^*\alpha \cup \pi_2^*\beta & & \\ & \searrow \tilde{\sigma} & \nearrow \cong & & \\ & & H^{2n+2}(S^2 \wedge S^{2n}) & & \\ & \nearrow & & & \\ H^{2n+3}(S^2 \vee S^{2n}) = 0 & & & & \end{array}$$

and $\sigma$ is just $\tilde{\sigma}(c_1\gamma_1 \otimes -)$ where $c_1\gamma_1 \in H^2S^2$ is the canonical generator.

Diagram (2) commutes because

$$ch(\beta x) = ch((\gamma_1 - 1)\otimes x) = ch(\gamma_1 - 1) \cup ch(x)$$

$$= (ch(\gamma_1) - 1) \cup ch\,x = (e^{c_1\gamma_1} - 1) \cup ch(x)$$

$$\overset{\boxed{b/c\ (c_1\gamma_1)^2 = 0}}{=} c_1\gamma_1 \cup ch(x) = \sigma(ch(x)).$$

It immediately follows that $ch : \tilde{K}^0(S^{2n+2}) \to \tilde{H}^{even}(S^{2n+2};\mathbb{Q})$ is injective, and $\sigma$ restricts to an isomorphism on the subgroups

$$\tilde{H}^{even}(-;\mathbb{Z}) \subset \tilde{H}^{even}(-;\mathbb{Q});$$

which shows that $ch(\tilde{K}^0 S^{2n+2}) = \tilde{H}^{even}(S^{2n+2};\mathbb{Z}) = H^{2n+2}(S^{2n+2};\mathbb{Z})$. $\square$

Note: There's no real need to use rational coeff's in this proof, because our computation showed that $ch(\beta(x))$ always lies in $\tilde{H}^{2n+2}(S^{2n+2};\mathbb{Z})$.

Corollary: For any sphere $S^k$, the Chern character gives an isomorphism $K^*(S^k)\otimes\mathbb{Q} \xrightarrow[ch]{\cong} H^*(S^k;\mathbb{Q})$.

Proof: For $k$ even, this is the prop'n. For $k$ odd, ($k=2n+1$) we have $K^0(S^{2n+1}) \cong K^0(S^{2n-1}) \cong \cdots \cong K^0(S^1) \cong \mathbb{Z}$, b/c complex v. bdles over $S^1$ are trivial (they're classified by maps $S^1 \to Gr_n \mathbb{C}^\infty = BU(n)$, and $\pi_1 BU(n) = \pi_0 U(n) = 0$).

So $ch: \underbrace{K^0 S^{2n+1}\otimes\mathbb{Q}}_{\mathbb{Q}} \to H^{even}(S^{2n+1};\mathbb{Q}) = H^0(S^{2n+1};\mathbb{Q}) = \mathbb{Q}$

is an isomorphism (it just records the dim'n of a trivial bdle).

Next, we defined $ch$ on $K^1$ by:

$$\begin{array}{ccc} K^1(S^{2n+1}) & \xrightarrow{ch} & \tilde{H}^{odd}(S^{2n+1};\mathbb{Q}) \\ \| & & \cong \\ \tilde{K}^0(S^{2n+2}) & \xrightarrow{ch} & \tilde{H}^{even}(S^{2n+2};\mathbb{Q}) \end{array}$$

So since the bottom map is an isom after tensoring w/ $\mathbb{Q}$, so is the top one. $\square$

We can now prove the main theorem.

<u>Theorem</u>: If $X$ is a finite CW cplx, then

$$Ch: K^*(X) \otimes \mathbb{Q} \longrightarrow H^*(X;\mathbb{Q})$$

is an isomorphism.

<u>Proof</u>: First, note that it suffices to prove that $\tilde{K}^0(SY)\otimes\mathbb{Q} \xrightarrow[ch]{\cong} \tilde{H}^{even}(SY;\mathbb{Q})$ for suspensions, because we have the diagram

$$\begin{array}{ccc} \tilde{K}^0(X) & \xrightarrow[\cong]{\beta} & \tilde{K}^0(S^2X) \\ \downarrow ch & & \downarrow ch \\ \tilde{H}^{even}(X;\mathbb{Q}) & \xrightarrow{\cong} & \tilde{H}^{even}(S^2X;\mathbb{Q}) \end{array}$$

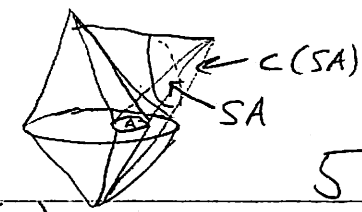
so if the RH vertical map is an isom., so is the left.

It then follows that the unreduced Chern character $K^0X \to H^{even}(X;\mathbb{Q})$ is an isom. as well. Finally, on $K^1$ we have

$$K^1(X)\otimes\mathbb{Q} \cong \tilde{K}^0(SX)\otimes\mathbb{Q} \xrightarrow{ch} \tilde{H}^{even}(SX) \cong H^{odd}(X)$$

so if $ch$ is an isom on $\tilde{K}^0(SX)$, it's an isom. on $K^1X$.

Now we prove the result by induction on the number of cells in $X$. When $X$ has one cell, $SX = *$ and the result holds. If $X$ has $n>1$ cells, then choose a subcomplex $A \subset X$ s.t. $X = A \cup e_n$, so $X/A = S^n$ and $SX/SA = S^{n+1}$. So by induction the result holds for $SA$ and for $SX/SA$. Now consider the




Puppe sequence

$$\begin{array}{ccccccccc} & & & & SX \cup C(SA) & \twoheadrightarrow & \frac{SX \cup C(SA)}{SX} & & \\ & & & \nearrow & & & & & \\ & & & & \cong & & \cong & & \\ X/A & \longrightarrow & SA & \longrightarrow & SX & \longrightarrow & SX/SA & \longrightarrow & S^2A. \\ \cong & & \cong & & \cong & & & & \\ X \cup CA & \subseteq & (X \cup CA) \cup CX & \twoheadrightarrow & \frac{(X\cup CA)\cup CX}{X \cup CA} & & & & \end{array}$$

Applying $\tilde{K}^0$ and the Chern character yields

$$\begin{array}{ccccccccc} \tilde{K}^0(S^2A) & \longrightarrow & \tilde{K}^0(SX/SA) & \longrightarrow & \tilde{K}^0(SX) & \longrightarrow & \tilde{K}^0(SA) & \to & \tilde{K}^0(X/A) \\ \downarrow ch & & \downarrow ch & & \downarrow ch & & ch \downarrow & & \downarrow ch \\ \tilde{H}^{even}(S^2A;\mathbb{Q}) & \longrightarrow & \tilde{H}^{even}(SX/SA;\mathbb{Q}) & \longrightarrow & \tilde{H}^{even}(SX;\mathbb{Q}) & \longrightarrow & \tilde{H}^{even}(SA;\mathbb{Q}) & \to & \tilde{H}^{even}(X/A;\mathbb{Q}). \end{array}$$

The bottom row may be constructed out of the Puppe sequence by applying $\tilde{H}^{even}$, ie. by piecing together the 3-term exact sequences of the various pairs. (This does produce the ordinary LES of the pair $(SA, SX)$, but we don't need to worry about that.)

Tensoring the top sequence with $\mathbb{Q}$ preserves exactness and makes the four outer vertical maps isom's; note that $S^2A \simeq \Sigma^2 A$ is a CW complex with the same number of cells as $A$.

$\square$