Since the RHS can be expressed in terms of the earlier Newton polynomials $s_0 = 1, s_1, \dots, s_{k-1}$, we obtain a recursive def'n for s_k, which we write in terms of new variables t_i (which we will substitute with $\sigma_i(x_1,\dots,x_n)$ shortly).

Def'n $$s_k(t_1,\dots,t_k) = \left(\sum_{i=1}^{k-1} (-1)^{k-i+1} s_i(t_1,\dots,t_i)\, t_{k-i}\right) + (-1)^{k+1} k t_k;$$

the recursion begins with $s_1(t_1) = 1$.

Lecture 15 We now need to prove the Fact:

Prop'n: The polynomials $s_k(t_1,\dots,t_k)$ defined above satisfy:

$$s_k(\sigma_1(x_1,\dots,x_n),\dots,\sigma_k(x_1,\dots,x_n)) = x_1^k + \dots + x_n^k$$

for *any* $n \geq 1$.

Pf: By induction.

For $k=1$, both sides are $\sum_{i=1}^{n} x_i$. Assuming the result for $l<k$, we have (when $n=k$)

$$s_k(\sigma_1(x_1,\dots,x_k),\dots,\sigma_k(x_1,\dots,x_k)) = \sum_{i=1}^{k-1} (-1)^{k-i+1} s_i(\sigma_1(x_1,\dots,x_k),\dots,\sigma_i(x_1,\dots,x_k))\,\sigma_{k-i}(x_1,\dots,x_k)$$
$$+ (-1)^{k-1} k\, \sigma_k(x_1,\dots,x_n)$$

$$\underset{\text{by induction}}{=} \left(\sum_{i=1}^{k-1} (-1)^{k-i+1} (x_1^i + \dots + x_k^i)\, \sigma_{k-i}(x_1,\dots,x_k)\right) + (-1)^{k+1} k \sigma_k(x_1,\dots,x_k)$$

$$= \sum_{i=0}^{k-1} (-1)^{k-i+1} (x_1^i + \dots + x_k^i)\, \sigma_{k-i}(x_1,\dots,x_k)$$

$$\underset{\text{by eq'n }(\star)}{=} x_1^k + \dots + x_k^k.$$

by eq'n ($\star$) i.e. by expanding $0 = \prod_j (x_i - x_j)$

So we get the desired formula, at least when $k=n$.

To prove the formula for $k>n$, we simply write

$$S_k(\sigma_1(x_1,\ldots,x_n),\ldots,\sigma_k(x_1,\ldots,x_n)) = S_k(\sigma_1(x_1,\ldots,x_n,\overbrace{0,\ldots,0}^{k-n}),\ldots,\sigma_k(x_1,\ldots,x_n,\overbrace{0,\ldots,0}^{k-n}))$$

$$\underset{\boxed{\text{by } k=n \text{ case}}}{=} x_1^k+\cdots+x_n^k+0^k+\cdots+0^k = x_1^k+\cdots+x_n^k$$

as desired.

On the other hand, say $k<n$. Then by the theory of symmetric poly's, we know that there exists a polynomial $S_{k,n}(t_1,\ldots,t_k)$ such that

$$(\star)\quad S_{k,n}(\sigma_1(x_1,\ldots,x_n),\ldots,\sigma_k(x_1,\ldots,x_n)) = x_1^k+\cdots+x_n^k.$$

Now

$$S_{k,n}(\sigma_1(x_1,\ldots,x_k),\ldots,\sigma_k(x_1,\ldots,x_k))$$

$$= S_{k,n}(\sigma_1(x_1,\ldots,x_k,\overbrace{0,\ldots,0}^{n-k}),\ldots,\sigma_k(x_1,\ldots,x_k,\overbrace{0,\ldots,0}^{n-k}))$$

$$= x_1^k+\cdots+x_k^k+0^k+\cdots+0^k = S_k(\sigma_1(x_1,\ldots,x_k),\ldots,\sigma_k(x_1,\ldots,x_k)).$$

But the theory of symmetric polynomials also says that there is a unique polynomial in k variables such that

$$p(\sigma_1(x_1,\ldots,x_k),\ldots,\sigma_k(x_1,\ldots,x_k)) = x_1^k+\cdots+x_k^k.$$

Since both $S_{k,n}$ and S_k satisfy this equation, we must have $S_k = S_{k,n}$, meaning that S_k satisfies $(\star)$. $\square$

We can now show that the Chern Character is a ring homomorphism:

<u>Theorem</u>: The Function

$$\mathbf{Vect}(X) \xrightarrow{Ch} H^*(X;\mathbb{Q})$$
$$[V] \longmapsto \dim V + \sum_{k>0} S_k(c_1 V, \ldots, c_k V)/k! \in \bigoplus_{i=0}^{\infty} H^{2i}(X;\mathbb{Q})$$

extends uniquely to a <u>ring</u> <u>homomorphism</u>

$$Ch: K^0(X) \longrightarrow H^*(X;\mathbb{Q}).$$

<u>Pf</u>: We are forced to define

$$Ch([V]-[W]) = \dim V - \dim W + \sum_{k>0} \frac{S_k(c_1 V, \ldots, c_k V) - S_k(c_1 W, \ldots, c_k W)}{k!},$$

and we must check that this is a ring homomorphism.

<u>Additivity</u>: We want to show that $ch(V \oplus W) = ch(V) + ch(W)$ (the formula on formal differences then follows formally).

We'll use the splitting principle; that is, we want a map $f: Y \to X$ s.t. $f^*: H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ is injective, and f^*V, f^*W are sums of line bdles. For integer coeff's, this is achieved by iterated use of projective bdles. The proof of the proj. Bdle Thm also works with $\mathbb{Q}$-coeff's, so $H^*(E;\mathbb{Q}) \to H^*(PE;\mathbb{Q})$ is always injective, But in fact (see HW 3) an injection $H^* X \hookrightarrow H^* Y$ w/ $\mathbb{Z}$-coeff's always gives an injection w/ $\mathbb{Q}$-coeff's.

Now, it suffices to show $ch(L_1 \oplus \cdots \oplus L_n) = \sum ch(L_i)$, for any line bdles $L_1, \ldots, L_n$. But this was exactly how we defined ch:

$$ch(L_1 \oplus \cdots \oplus L_n) = n + \sum_{k>0} \frac{s_k(c_1(L_1 \oplus \cdots \oplus L_n), \ldots, c_k(L_1 \oplus \cdots \oplus L_n))}{k!}$$

$$= n + \sum_{k>0} \frac{s_k(\sigma_1(c_1 L_1, \ldots, c_1 L_n), \ldots, \sigma_k(c_1 L_1, \ldots, c_1 L_n))}{k!}$$

$$= n + \sum_{k>0} \frac{(c_1 L_1)^k + \cdots + (c_1 L_n)^k}{k!} = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{(c_1 L_i)^k}{k!}$$

$$= \sum_{i=1}^{n} e^{c_1 L_i}.$$

We originally defined $Ch(L_i) = e^{c_1 L_i}$ for line bdles, but we do need to check that this is consistent with our general def'n, i.e. that $\displaystyle\sum_{k=0}^{\infty} \frac{(c_1 L)^k}{k!} = \sum_{k=0}^{\infty} \frac{s_k(c_1 L, \overbrace{c_2 L, \ldots, c_k L}^{\boxed{\text{all zero}}})}{k!}$

<u>Claim</u>: For each k, $s_k(t, 0, \ldots, 0) = t^k$.

<u>Pf</u>: By induction on k, using the recursion relation.

For $k = 0$, $s_0 = 1 = (c_1 L)^0$. Assuming the Claim for $l < k$, we have

$t_{k-i} = 0$ for $i < k-1$

$$s_k(t, 0, \ldots, 0) = (-1)^{k+1} k \cdot 0 + \sum_{i=1}^{k-2} (-1)^{k-i+1} s_i(t, 0, \ldots, 0) \cdot 0 + \cancel{(-1)^{k-(k-1)+1}} s_{k-1}(t, 0, \ldots, 0) t$$

$$\overset{\boxed{\text{induction}}}{=} (t^{k-1}) \cdot t = t^k. \quad \square$$

<u>Multiplicativity</u>: We need to show that

$$\mathrm{Ch}(V \otimes W) = \mathrm{Ch}(V)\,\mathrm{Ch}(W).$$

We again can check the case of sums of line bdles, and then apply the splitting principle.

$$\mathrm{Ch}((L_1 \oplus \cdots \oplus L_n) \otimes (L_1' \oplus \cdots \oplus L_m')) = \mathrm{Ch}\Big(\bigoplus_{i,j} L_i \otimes L_j'\Big)$$

$$\overset{\boxed{\text{Additivity}}}{=} \sum_{i,j} \mathrm{Ch}(L_i \otimes L_j') = \sum_{i,j} e^{c_1(L_i \otimes L_j')}$$

$$= \sum_{i,j} e^{c_1 L_i + c_1 L_j'} = \sum_{i,j} e^{c_1 L_i} e^{c_1 L_j'}.$$

On the other hand,

$$\mathrm{Ch}(L_1 \oplus \cdots \oplus L_n)\,\mathrm{Ch}(L_1' \oplus \cdots \oplus L_m') = \Big(\sum_i \mathrm{Ch}\,L_i\Big)\Big(\sum_j \mathrm{Ch}(L_j')\Big)$$

$$= \sum_{i,j} (\mathrm{Ch}\,L_i)(\mathrm{Ch}\,L_j') = \sum_{i,j} e^{c_1 L_i} e^{c_1 L_j'}. \quad \square$$

We can now extend Ch to a map (an additive hom., at least)

$$K^1 X = \tilde{K}^0(SX) \longrightarrow \bigoplus_{i=0}^{\infty} H^{2i+1}(X;\mathbb{Q}) \subset H^*(X;\mathbb{Q}).$$

This is achieved by the diagram

$$\begin{tikzcd} K^0(SX) \arrow[r, "\mathrm{Ch}"] \arrow[d] & \bigoplus H^{2i}(SX;\mathbb{Q}) \arrow[d] \\ K^0(pt) \arrow[r, "\mathrm{Ch}"] & \bigoplus H^{2i}(pt;\mathbb{Q}), \end{tikzcd}$$

which (upon taking kernels) yields the Chern Character

$$K^1 X := \tilde{K}^0(SX) \longrightarrow \bigoplus_{i \geq 0} \tilde{H}^{2i}(SX;\mathbb{Q}) \cong \bigoplus_{i \geq 0} H^{2i-1}(X;\mathbb{Q}).$$

Our goal ~~is to~~ show that Ch is an isomorphism whenever X is a finite CW cplx. We need two basic facts from K-theory:

1) Exact Sequence (Hatcher VB Prop. 2.9)
If $A \subset X$ is closed (X cpt Hausdorff) then the sequence $A \xrightarrow{i} X \xrightarrow{q} X/A$ induces an exact sequence

$$\tilde{K}^0(X/A) \xrightarrow{q^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(A)$$

2) Bott Periodicity (Hatcher VB Thm 2.11)

There are natural isomorphisms

$$\beta : \tilde{K}^0(X) \xrightarrow{\cong} \tilde{K}^0(\underbrace{S^2 X}_{\text{(unreduced) double suspension}})$$

for all compact Hausdorff space X. The map β is essentially tensor product with $[\gamma_1^1]-1$, where γ_1^1 is the tautological bdle over $S^2 \cong \mathbb{C}P^1$.

2) will allow us to compute Ch for spheres; and using 1) we can extend to all finite CW complexes.

Proof of 1 (Sketch):

- $i^* q^* = (qi)^*$, but qi is constant so $(qi)^*$ is 0 on $\tilde{K}^0$.
- To show that $\mathrm{Ker}(i^*) \subseteq \mathrm{Im}(q^*)$, first note that $\tilde{K}^0(Z)$ is just $\mathrm{Vect}(Z)/_{\text{stable isom.}}$, so we

just want to show that if i^*E is stably trivial, then i^*E is stably isomorphic to some q^*V.

Since $i^*E = E|_A$ is stably trivial, we have $(E\oplus\varepsilon^n)|_A \cong \varepsilon^m$ for some n and m. We'll show that since this bdle is trivial over A, it is in fact the pullback of a bdle over X/A.

In fact, say E is a bdle over X which is trivial over A. Fixing a trivialization φ over A, we can identify the various fibers over A to form a quotient space $\begin{array}{c}\bar{E} \\ \downarrow \\ X/A\end{array} = E/\varphi^{-1}(a,v)\sim\varphi^{-1}(a',v)$.

Topologically, we have

$$\begin{tikzcd} E \arrow[r, "\tilde{q}"] \arrow[d] & \bar{E} \arrow[d] \\ X \arrow[r, "q"] & X/A \end{tikzcd}$$

and $\tilde{q}$ is a continuous fiber-wise isomorphism.

It remains only to check that $\begin{array}{c}\bar{E} \\ \downarrow \\ X/A\end{array}$ is locally trivial on a nbd of the basept $A/A \in X/A$.

If A is a subcomplex of a CW cplx X, then one has a nbdh $U\supset A$ which deformation retracts to A. Then the BHT $\Rightarrow$ $E|_U$ is isomorphic to $r^*E|_A \cong \varepsilon^m$ (r the retraction $U\to A$).

Now the trivialization of $E|_U$ descends to a trivialization of $\bar{E}|_{U/A}$; note that U/A is open in X/A. (Check: $U\times\mathbb{C}^n/_{(a,v)\sim(a',v)} \cong U/A\times\mathbb{C}^n$)

We need to understand the Bott Periodicity map

$$\beta : \tilde{K}^0 X \longrightarrow \tilde{K}^0(S^2 X).$$

$\uparrow$ unreduced suspension

This will be the composite

$$\begin{array}{ccccc} \tilde{K}^0 X & \longrightarrow & \tilde{K}^0(S^2) \otimes \tilde{K}^0 X & \xrightarrow{\tilde{\mu}} & \tilde{K}^0(S^2 X) \\ [\alpha] & \longmapsto & [\gamma_1^1] \otimes [\alpha] & & \end{array}$$

where $\tilde{\mu}$ is built out of the map

$$\begin{array}{ccc} K^0 X \otimes K^0 Y & \xrightarrow{*} & K^0(X \times Y) \\ [V] \otimes [W] & \longmapsto & [\pi_1^* V \otimes \pi_2^* W]. \end{array}$$

We need to replace $S^2 \times X$ with $S^2 X \simeq S^2 \wedge X = S^2 \times X / S^2 \vee X$

$\uparrow$ reduced suspension

<u>Claim</u>: 1) The image under $*$ of $\tilde{K}^0(X) \otimes K^0(Y)$ lies in $\mathrm{Im}(\tilde{K}^0(X \wedge Y) \longrightarrow \tilde{K}^0(X \times Y))$.

2) The map $\tilde{K}^0(X \wedge Y) \xrightarrow{\pi^*} \tilde{K}^0(X \times Y)$ is <u>injective</u>.

We can now define

$$\tilde{K}^0(X) \otimes \tilde{K}(Y) \xrightarrow{\tilde{\mu}} \tilde{K}^0(X \wedge Y)$$

to simply be the map $* : K^0(X) \otimes K^0(Y) \longrightarrow K^0(X \times Y)$ restricted to $\tilde{K}^0(X) \otimes \tilde{K}^0(Y)$ and considered as a map into $\tilde{K}^0(X \wedge Y) \hookrightarrow \tilde{K}^0(X \times Y)$.

<u>Proof of Claim:</u> First, note that $\tilde{K}^0(X) \otimes \tilde{K}^0(Y) \hookrightarrow K^0X \otimes K^0Y \cancel{\to K^0(X \times Y)}$ has image in $\tilde{K}^0(X \times Y)$ (by a simple calculation).

We have the short exact sequence

$$\tilde{K}^0(X \wedge Y) \xrightarrow{q^*} \tilde{K}^0(X \times Y) \xrightarrow{i^*} \tilde{K}^0(X \vee Y)$$

so to show that $\mathrm{Im}(\tilde{K}^0(X) \otimes \tilde{K}^0(Y) \to \tilde{K}^0(X \times Y))$ lands in $\mathrm{Im}\, q^*$, we just need to check that $i^*(\alpha_1 * \alpha_2) = 0 \in \tilde{K}^0(X \vee Y)$. Note that the maps

$$X \leftrightarrows X \vee Y \leftrightarrows Y = (X \vee Y)/X$$

give splittings of

$$\tilde{K}^0(Y) \to \tilde{K}^0(X \vee Y) \to \tilde{K}^0(X),$$

so the restriction maps
$$i_X^* : \tilde{K}^0(X \vee Y) \to \tilde{K}^0(X)$$
$$i_Y^* : \tilde{K}^0(X \vee Y) \to \tilde{K}^0(Y)$$

give an isomorphism $\tilde{K}^0(X \vee Y) \xrightarrow[\cong]{i_X^* \oplus i_Y^*} \tilde{K}^0(X) \oplus \tilde{K}^0(Y)$.

So $i^*(\alpha_1 * \alpha_2) = 0 \in \tilde{K}^0(X \vee Y) \iff$ the restrictions of $i^*(\alpha_1 * \alpha_2)$ to X and to Y are both trivial. But we have a diagram

$$\begin{tikzcd}
\tilde{K}^0(X) \otimes \tilde{K}^0(Y) \arrow[r, "*"] \arrow[d] & \tilde{K}^0(X \times Y) \arrow[d] \arrow[r] & \tilde{K}^0(X \vee Y) \arrow[dl] \\
\tilde{K}^0(X) \otimes \underbrace{\tilde{K}^0(y_0)}_{\text{trivial group}} \arrow[r] & \tilde{K}^0(X \times \{y_0\}) &
\end{tikzcd}$$

$\leftarrow$ basept in Y

which gives $i^*(\alpha_1, \alpha_2)|_X = 0$, and similarly for Y.

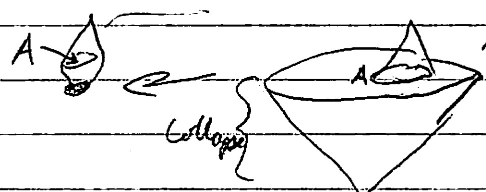
To prove 2), we need to extend the SES

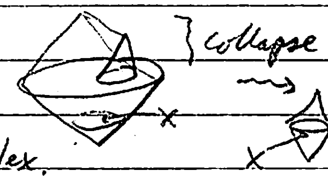
$$\tilde{K}^0(X\wedge Y) \to \tilde{K}^0(X\times Y) \to \tilde{K}^0(X\vee Y)$$

to the left. This is done via the "Puppe Sequence"

$$A \hookrightarrow X \to X/A \simeq X\cup CA \hookrightarrow \underbrace{X\cup CA\cup CX}_{\cong\, SA} \to SX \simeq (X\cup CA\cup CX)\cup C(X\cup CA)$$

collapse $C(X\cup CA)$





The collapse maps are htpy equiv's if $A\subset X$ is a subcomplex.
Each 3-term piece is of the form $B \hookrightarrow Z \hookrightarrow B/Z$ so we

SES associated to $X\cup CA \hookrightarrow X\cup CA\cup CX \to \frac{X\cup CA\cup CX}{X\cup CA} \cong SX$

$$\begin{tikzcd}
\tilde{K}^0\left(\frac{X\cup CA\cup CX}{X\cup CA}\right) \arrow[r] & \tilde{K}^0(X\cup CA\cup CX) \arrow[r] & \tilde{K}^0(X\cup CA) & & \\
\tilde{K}^0(SX) \arrow[r] \arrow[u, "\cong"] & \tilde{K}^0(SA) \arrow[r] \arrow[u, "\cong"] \arrow[d, equal] & \tilde{K}^0(X/A) \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"] & \tilde{K}^0(X) \arrow[r] & \tilde{K}^0(A) \\
& \tilde{K}^0((X\cup CA)/X) \arrow[r] & \tilde{K}^0(X\cup CA) \arrow[ur] & &
\end{tikzcd}$$

SES for $X \hookrightarrow X\cup CA \to (X\cup CA)/X \cong SA$

Specializing, we have

$$\begin{tikzcd}
\tilde{K}^0(S(X\times Y)) \arrow[r] & \tilde{K}^0(S(X\vee Y)) \arrow[r, "0"] \arrow[d, equal] & \tilde{K}^0(X\wedge Y) \arrow[r] & \tilde{K}^0(X\times Y) \arrow[r] & \tilde{K}^0(X\vee Y) \\
& \tilde{K}^0(SX\vee SY) \arrow[d, equal] & & & \\
& \tilde{K}^0(SX)\oplus\tilde{K}^0(SY) \arrow[uul, "\text{splitting}"] & & &
\end{tikzcd}$$

The splitting $\Rightarrow \tilde{K}^0(S(X\vee Y)) \xrightarrow{0} \tilde{K}^0(X\wedge Y)$
$\Rightarrow \tilde{K}^0(X\wedge Y) \hookrightarrow \tilde{K}^0(X\times Y)$
as desired. $\square$