# Lecture 14

## K-theory and The Chern Character:

Def'n: For any topological space $X$, we define

$$K^0(X) = Gr(\underline{Vect}(X))$$

where $\underline{Vect}(X)$ denotes the commutative monoid of isomorphism classes of cplx vector bdles over $X$, and $Gr(-)$ denotes group completion, i.e. the Grothendieck construction. The monoid structure is given by Whitney sum.

By def'n, $K^0(X) = \underline{Vect}(X) \times \underline{Vect}(X) / \sim$, where the equivalence rel'n $\sim$ is:

$$([V],[W]) \sim ([V'],[W']) \text{ if } V \oplus W' \cong V' \oplus W.$$

We think of $([V],[W]) \in K^0(X)$ as the formal difference b/w $[V]$ and $[W]$, and write this el't as

$$[V]-[W].$$

Then $[V]-[W] = [V']-[W'] \iff V \oplus W' \cong V' \oplus W,$
$\iff [V]+[W'] = [V']+[W]$ in $\underline{Vect}(X)$.

__Universal Property:__

If $\underline{\text{Vect}}(X) \xrightarrow{f} G$ is any homomorphism from $\underline{\text{Vect}}(X)$ to an abelian group $A$, then there exists a unique extension of $f$ to $K^0(X)$:

$$\begin{tikzcd} \underline{\text{Vect}}(X) \arrow[r, "f"] \arrow[dr] & A \\ & K^0(X) \arrow[u, dashed, "\exists \tilde{f}"'] \end{tikzcd}$$

$$[V] \longmapsto [V]-[X]$$

← the unique 0-dim'l bdle $\begin{matrix} X \\ \downarrow \\ X \end{matrix}$

__Pf:__ Since $\tilde{f}$ must be a homomorphism, we must have

$$\tilde{f}\big([V]-[W]\big) = \tilde{f}[V] - \tilde{f}[W] = f[V] - f[W].$$

inverses exist in $A$

So $\tilde{f}$ is unique, and this formula does give a well-defined homomorphism. $\square$

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We can also define $K^1(X)$:

__Def'n:__ $K^1(X) = \tilde{K}^0(SX)$, where $S(X) = X \times I / {X \times 0, \atop X \times 1}$ is the (unreduced) suspension of $X$, and

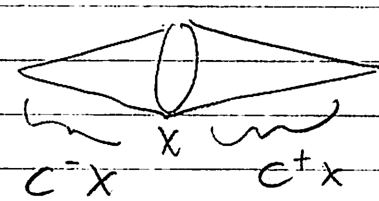
$$\tilde{K}^0(Z) := \ker\big(K^0 X \to K^0(x_0)\big)$$

Note: $K^0(pt) = \mathbb{Z}$, b/c $\mathrm{Vect}(pt) = \mathbb{N}$.

The map $K^0(X) \longrightarrow K^0(x_0) \cong \mathbb{Z}$ just sends $[V]-[W]$ to $\dim V - \dim W$, so is defined independently of $x_0 \in X$.

Relation to Banach Algebras:

Vector bdles $\begin{array}{c} E \\ \downarrow \\ SX \end{array}$ are automatically trivial over the two cones $C^+X, C^-X \subseteq SX$:




Hence bdles over $SX$ are determined by a clutching function $X \longrightarrow GL_n \mathbb{C}$, i.e. an element in

$$GL_n(\underbrace{C^\circ X}_{\text{cts complex-valued fcns on } X})$$

Theorem:
For any finite CW cplx $X$, there are isomorphisms of $\mathbb{Q}$-vector spaces

$$\bigoplus_{i=0}^{\infty} H^{2i}(X;\mathbb{Q}) \cong K^0(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\bigoplus_{i=0}^{\infty} H^{2i+1}(X;\mathbb{Q}) \cong K^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular, $\mathrm{Rank}(K^0X) = \sum_{i=0}^{\infty} \mathrm{Rank}\, H^{2i}(X;\mathbb{Z})$

$\mathrm{Rank}(K^1X) = \sum_0^{\infty} \mathrm{Rk}(H^{2i+1}(X;\mathbb{Z}))$.

In fact, there is a ring structure on $K^*X = K^0X \oplus K^1X$ such that the above maps give a ring isomorphism

$$K^*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^*(X;\mathbb{Q}).$$

This isomorphism is known as the Chern Character, and is defined in terms of the Chern classes of $[V] \in K^*(X)$.

Ring Structure on $K^*X$:

The ring structure is graded, so

$$K^0X \otimes K^0X \longrightarrow K^0X, \quad K^1X \otimes K^0X \to K^1X, \text{ etc.}$$

On $K^0X$, it is defined simply by tensor product:

$$([V]-[W]) \cdot ([V']-[W'])$$
$$= [V \otimes V'] - [W \otimes V'] - [V \otimes W'] + [W \otimes W'].$$

We'll more or less ignore the rest of the ring structure, which is harder to describe.

To begin, we'll construct the Chern Character

$$Ch: K^0(X) \longrightarrow \bigoplus_{i=0}^{\infty} H^{2i}(X;\mathbb{Q}).$$

We want
$$ch(E \oplus F) = ch(E) + ch(F)$$
$$ch(E \otimes F) = ch(E)\, ch(F).$$

By the Splitting Principle, we expect that it will suffice to define $ch(L_1 \oplus \cdots \oplus L_n)$ for $L_i$ line bdles.

<u>Idea</u>: For line bdles,
$$c_1(L_1 \otimes L_2) = c_1 L_1 + c_1 L_2$$
and we need to switch the operations $+$ and $\cdot$ on the right. This is achieved by exponentiating:

<u>Def'n</u>: For a line bdle $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ (with $X$ a <u>finite</u> CW cplx) we set
$$ch(L) = e^{c_1 L} = 1 + c_1 L + (c_1 L)^2/2! + (c_1 L)^3/3! + \cdots$$
$$\in H^*(X; \mathbb{Q}).$$

(Here $c_1 L$ denotes the image of $c_1 L \in H^2(X; \mathbb{Z})$ under the map $H^2(X;\mathbb{Z}) \longrightarrow H^2(X;\mathbb{Q})$ induced by $\mathbb{Z} \hookrightarrow \mathbb{Q}$.)

<u>Note</u>: The sum defining $e^{c_1 L}$ is finite b/c $X$ is finite dim'l.

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If $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bdles, then we <u>must</u> set $ch(E) = \sum_{i=1}^{n} ch(L_i) = \left(1 + c_1 L_1 + (c_1 L_1)^2/2 + \cdots\right) + \cdots + \left(1 + c_1 L_n + (c_1 L_n)^2/2 + \cdots\right)$

We want to reinterpret this formula in such a way that it extends to all vector bdles $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$.

Grouping terms in the previous formula gives

$$ch(E) = n + (c_1L_1 + \cdots + c_1L_n) + \frac{(c_1L_1)^2 + \cdots + (c_1L_n)^2}{2!} + \cdots + \frac{(c_1L_1)^k + \cdots + (c_1L_n)^k}{k!} + \cdots$$

On the other hand, the total Chern class of $E$ is

$$c(E) = c(L_1) \cdots c(L_n) = (1 + c_1L_1) \cdots (1 + c_1L_n)$$

$$= 1 + \underbrace{\sigma_1(c_1L_1, \dots, c_1L_n)}_{c_1E} + \cdots + \underbrace{\sigma_n(c_1L_1, \dots, c_1L_n)}_{c_nE}$$

where $\sigma_i$ denotes the $i^{th}$ elementary symmetric polynomial:

$$\sigma_i(x_1, \dots, x_n) = \sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} \cdots x_{j_i} .$$

Fact: There exist polynomials $S_k(x_1, \dots, x_k)$ $(k = 1, 2, \dots)$ such that $\boxed{\text{For any } n}$, $S_k(\sigma_1(x_1, \dots, x_n), \dots, \sigma_k(x_1, \dots, x_n)) = x_1^k + \cdots + x_n^k$

Although a priori, there could be a dependence on n.

These are called the Newton Polynomials, and their existence is guaranteed by the basic theory of symmetric polynomials: any symm. poly of degree $k$ (e.g. $x_1^k + \cdots + x_n^k$) is a poly in $\sigma_1(x_1, \dots, x_n), \dots, \sigma_k(x_1, \dots, x_n)$. We'll construct $S_k$ explicitly.

We now have $S_k(\overbrace{\sigma_1(c_1L_1, \dots, c_1L_n)}^{c_1E}, \dots, \overbrace{\sigma_n(c_1L_1, \dots, c_1L_n)}^{c_nE}) = (c_1L_1)^k + \cdots + (c_1L_n)^k$

So $ch(E) = \overset{\boxed{n = \dim E}}{n} + \sum c_1L_i + \frac{\sum (c_1L_i)^2}{2!} + \cdots = n + S_1(c_1E) + S_2(c_1E, c_2E)/2! + \cdots$

<u>Def'n</u>: For any vector bdle $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ w/ $X$ fin. dim'l,

$$ch(E) = \dim(E) + \sum_{k>0} s_k(c_1E, \dots, c_kE)/k! \in H^{*}(X;\mathbb{Q})$$

What are the Newton Polynomials?

- $s_1(x) = x$; so $s_1(\sigma_1(x_1, \dots, x_n)) = s_1(x_1 + \dots + x_n) = x_1 + \dots + x_n$, as desired.
- $s_2(x_1, x_2) = x_1^2 - 2x_2$;

so $s_2(\sigma_1(x_1, \dots, x_n), \sigma_2(x_1, \dots, x_n)) = (x_1 + \dots + x_n)^2 - 2\sum_{i<j} x_i x_j$

$$= x_1^2 + \dots + x_n^2 + \sum_{i \neq j} x_i x_j - 2\sum_{i<j} x_i x_j$$

$$= x_1^2 + \dots + x_n^2,$$

as desired.

To construct $s_k$ in general, we start with the following observation:

$$(t - x_1) \cdots (t - x_k) = \sum_{i=0}^{k} (-1)^{k-i} t^i \sigma_{k-i}(x_1, \dots, x_k),$$

and setting $t = x_j$ yields

$$0 = \sum_{i=0}^{k} (-1)^{k-i} x_j^i \sigma_{k-i}(x_1, \dots, x_k)$$

i.e.

$$x_j^k = \sum_{i=0}^{k-1} (-1)^{k-i+1} x_j^i \sigma_{k-i}(x_1, \dots, x_k)$$

and hence (summing over $j = 1, \dots, k$)

$$(\star) \quad x_1^k + \dots + x_k^k = \sum_{i=0}^{k-1} (-1)^{k-i+1} (x_1^i + \dots + x_k^i) \sigma_{k-i}(x_1, \dots, x_k)$$

Since the RHS can be expressed in terms of the earlier Newton polynomials $s_0 = 1, s_1, \dots, s_{k-1}$, we obtain a recursive def'n for $s_k$, which we write in terms of new variables $t_i$ (which we will substitute with $\sigma_i(x_1, \dots, x_n)$ shortly).

Def'n $s_k(t_1, \dots, t_k) = \left(\sum_{i=1}^{k-1} (-1)^{k-i+1} s_i(t_1, \dots, t_i)\, t_{k-i}\right) + (-1)^{k+1} k t_k$; the recursion begins with $s_1(t_1) = 1$.

Lecture 15 We now need to prove the Fact:

Prop'n: The polynomials $s_k(t_1, \dots, t_k)$ defined above satisfy:

$$s_k(\sigma_1(x_1, \dots, x_n), \dots, \sigma_k(x_1, \dots, x_n)) = x_1^k + \dots + x_n^k$$

for any $n \geq 1$.

Pf: By induction.

For $k = 1$, both sides are $\sum_{i=1}^{n} x_i$. Assuming the result for $l < k$, we have (when $n = k$)

$$s_k(\sigma_1(x_1, \dots, x_k), \dots, \sigma_k(x_1, \dots, x_k)) = \sum_{i=1}^{k-1} (-1)^{k-i+1} s_i(\sigma_1(x_1, \dots, x_k), \dots, \sigma_i(x_1, \dots, x_k))\, \sigma_{k-i}(x_1, \dots, x_k)$$

$$+ (-1)^{k-1} k\, \sigma_k(x_1, \dots, x_n)$$

$$\underset{\text{by induction}}{=} \left(\sum_{i=1}^{k-1} (-1)^{k-i+1} (x_1^i + \dots + x_k^i)\, \sigma_{k-i}(x_1, \dots, x_k)\right) + (-1)^{k+1} k\, \sigma_k(x_1, \dots, x_k)$$

$$= \sum_{i=0}^{k-1} (-1)^{k-i+1} (x_1^i + \dots + x_k^i)\, \sigma_{k-i}(x_1, \dots, x_k)$$

$$\underset{\text{by eq'n } (\star)}{=} x_1^k + \dots + x_k^k.$$

So we get the desired formula, at least when $k = n$.