Applications of Stiefel-Whitney Classes

Thm [MS 4.8]: If $\mathbb{R}P^{2^r}$ can be immersed in $\mathbb{R}^{2^r+k}$, then $k \geq 2^r - 1$.

We'll prove this by studying the Stiefel-Whitney classes of $T(\mathbb{R}P^n)$, the tangent bundle. First, recall:

Def'n: If M, N are smooth mflds, an immersion $f: M \to N$ is a smooth map whose derivative $Df: TM \to TN$ is injective at every point.

An immersion $M \to \mathbb{R}^n$ puts restrictions on TM:

Lemma: If $f: M \to N$ is an immersion, then there is a direct sum decomposition

$$f^*(TN) \cong TM \oplus E$$

for some $\begin{array}{c} E \\ \downarrow \\ M \end{array}$.

Pf: We have a diagram

$$\begin{tikzcd} TM \arrow[r, dashed] \arrow[rr, bend left, "Df"] \arrow[dr] & f^*TN \arrow[r] \arrow[d] & TN \arrow[d] \\ & M \arrow[r, "f"] & N \end{tikzcd}$$

So there is an induced map $TM \to f^*TN$ (by the universal property of pullbacks). On each fiber we have

$$\begin{tikzcd} T_xM \arrow[r, hook, "D_xf"] \arrow[dr] & T_xN \\ & (f^*TN)_x \arrow[u, "\cong"'] \end{tikzcd}$$

so $T_xM \hookrightarrow (f^*TN)_x$ is injective.

Now if we put a metric on f^*TN, the subbundle T_xM has an orthogonal complement E as desired. □

Corollary: If $\underbrace{M^n}_{\text{n-dim'l mfld}} \xrightarrow{f} \mathbb{R}^{n+k}$ is an immersion, then $\omega(M) := \omega(TM)$ is a unit in the ring $H^*(M;\mathbb{Z}/2)$, and its inverse $\bar{\omega}(M)$ lies in $\bigoplus_{i=0}^{k} H^i(M;\mathbb{Z}/2)$

Proof: We have $TM \oplus E \cong f^*T(\mathbb{R}^n) \cong M \times \mathbb{R}^n$, so by the Whitney Sum Formula

$$\omega(TM) \cdot \omega(E) = \omega(M \times \mathbb{R}^n) = 1. \quad \square$$

Notice: If $x \in H^*(M;\mathbb{Z}/2) = \bigoplus_{i=0}^{n} H^n(M;\mathbb{Z}/2)$ is a class whose 0-dim'l component is 1 (e.g. any total Stiefel-Whitney class) then one can always solve the system of equations

$$(1+x_1+ - +x_n) \cdot (1+a_1+a_2+a_3+\cdots+a_n) = 1$$

(where $x = 1+x_1+\cdots+x_n$ and $x_i, a_i \in H^i(M;\mathbb{Z}/2)$).

So the interesting part of the Corollary is that the inverse lies in $\bigoplus_{i=0}^{k} H^i(M;\mathbb{Z}/2)$.

If $k \geq n$, then this condition is vacuous, and in fact Whitney's Immersion Thm says an n-Mfld always immerses in $\mathbb{R}^{2n-1}$.

So we need to <u>compute</u> $w(T\mathbb{RP}^n)$. The key is that we can describe $T\mathbb{RP}^n$ in terms of the tautological line bdle $\begin{matrix}\gamma_n^1\\ \downarrow\\ \mathbb{RP}^n\end{matrix}$.

<u>Lemma</u> [MS 4.4]: $T(\mathbb{RP}^n) \cong \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp)$, where $(\gamma_n^1)^\perp$ is the orthogonal complement of γ_n^1 inside $\mathbb{RP}^n \times \mathbb{R}^{n+1}$.

(<u>Note</u>: By def'n, $\gamma_n^1 \subseteq \mathbb{RP}^n \times \mathbb{R}^{n+1}$.)

<u>Pf</u>: Since the quotient map $S^n \xrightarrow{q} \mathbb{RP}^n$ is smooth and $D_x q : T_x S^n \longrightarrow T_{[\pm x]} \mathbb{RP}^n$ is always an isomorphism, we can identify $T\mathbb{RP}^n$ with $TS^n / v \sim (D\alpha)v$ where $\alpha : S^n \to S^n$ is the antipodal map ($\alpha(x) = -x$).

Writing $TS^n = \{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1,\ x \cdot v = 0\}$, we have $T\mathbb{RP}^n = \{(x, v) \mid \|x\| = 1,\ x \cdot v = 0\} / (x, v) \sim (-x, -v)$.

But there is a linear isomorphism

$$\{(x,v) \mid \|x\| = 1,\ x \cdot v = 0\}/_{(x,v)\sim(-x,-v)} \xrightarrow{\cong} \mathrm{Hom}(\mathrm{Span}(x), \mathrm{Span}(x)^\perp)$$

$$\{(x,v), (-x,-v)\} \longrightarrow \ell : \mathrm{Span}(x) \longrightarrow \mathrm{Span}(x)^\perp$$

$$x \longmapsto v$$
$$-x \longmapsto -v$$

(with inverse $\ell \longmapsto \{(u, \ell(u)), (-u, \ell(-u))\}$ where $u \in \mathrm{Span}(x)$ is a unit vector – i.e. $u = \pm x$).

These isomorphisms give a cont. linear isom.

$$T\mathbb{R}P^n \xrightarrow{\cong} \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp). \qquad \square$$

We can now compute the Stiefel-Whitney classes of $T(\mathbb{R}P^n)$:

<u>Theorem</u> (MS 4.5): $w(\mathbb{R}P^n) = (1 + w_1\gamma_n^1)^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} (w_1\gamma_n^1)^i$.

(Recall that $w_1(\gamma_n^1) \in H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ is the canonical generator of the ring $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$, which we've been denoting by α.
So $w(\mathbb{R}P^n) = (1+\alpha)^{n+1}$.)

<u>Proof</u>: We claim that

$$T(\mathbb{R}P^n) \oplus \underset{\substack{\uparrow \\ \text{trivial line bdle}}}{\varepsilon^1} \cong \underbrace{\gamma_n^1 \oplus \cdots \oplus \gamma_n^1}_{n+1}.$$

The Whitney Sum Formula (together with Exercise 4 on HW2) then completes the proof.

We have:

$$T(\mathbb{R}P^n) \oplus \varepsilon^1 \cong \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp) \oplus \overbrace{\mathrm{Hom}(\gamma_n^1, \gamma_n^1)}^{\text{line bdle w/ non-zero section, hence trivial}}$$

$$\cong \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp \oplus \gamma_n^1)$$

$$\underset{\substack{\text{by defn of} \\ (\gamma_n^1)^\perp}}{\cong} \mathrm{Hom}(\gamma_n^1, \underset{\substack{\uparrow \\ \text{trivial } (n+1)\text{-plane bdle}}}{\varepsilon^{n+1}}) \cong \underbrace{(\gamma_n^1)^*}_{\text{dual bdle}} \oplus \cdots \oplus (\gamma_n^1)^*$$

$$\cong \gamma_n^1 \oplus \cdots \oplus \gamma_n^1. \qquad \square$$

Now we can prove our immersion result:

Thm [MS 4.8] If $\mathbb{R}P^{2^r}$ can be immersed in $\mathbb{R}^{2^r+k}$, then $k \geq 2^r - 1$.

Pf: We begin by computing $w(\mathbb{R}P^{2^r})$:

$$w(\mathbb{R}P^{2^r}) = (1+\alpha)^{2^r+1} = (1+\alpha)(1+\alpha)^{2^r}$$

$$= (1+\alpha)(1+\alpha)^2 \cdots (1+\alpha)^2$$

$$= (1+\alpha)(1+\cancel{2\alpha}+\alpha^2)\cdots(1+\cancel{2\alpha}+\alpha^2)$$

$$\overset{\mathbb{Z}/2 \text{ coeff's}}{=} (1+\alpha)(1+\alpha^2)^{2^{r-1}} = \cdots = (1+\alpha)(1+\alpha^{2^r})$$

$$= 1+\alpha+\alpha^{2^r}+\underbrace{\alpha^{2^r+1}}_{0,\ \text{for dim'l reasons}} = \boxed{1+\alpha+\alpha^{2^r}}.$$

Next, we compute the inverse of $w(\mathbb{R}P^{2^r})$ in $H^*(\mathbb{R}P^{2^r};\mathbb{Z}/2)$:

Claim: $(1+\alpha+\alpha^{2^r})(1+\alpha+\alpha^2+\cdots+\alpha^{2^r-1}) = 1$

Pf: Expanding gives:

$$(1+\alpha+\alpha^2+\cdots+\alpha^{2^r-1}) + (\alpha+\alpha^2+\cdots+\alpha^{2^r}) + (\alpha^{2^r}+0+\cdots+0)$$

$$= 1+2\alpha+2\alpha^2+\cdots+2\alpha^{2^r} = 1. \qquad \square$$

Now, by the Corollary (p. 2) we know that if $\mathbb{R}P^{2^r}$ immerses in $\mathbb{R}^{2^r+k}$, then

$$\bar{\omega}(\mathbb{R}P^{2^r}) := (\omega(\mathbb{R}P^{2^r}))^{-1}$$

lies in $\bigoplus_{i=0}^{k} H^i(M; \mathbb{Z}/2)$. But

$$\bar{\omega}(\mathbb{R}P^{2^r}) = 1 + \alpha + \cdots + \alpha^{2^r-1},$$

so $k \geq 2^r - 1$ as claimed. $\square$

Rmk: This shows that Whitney's Immersion Thm is the best possible when the dim'n of the mfld is a power of 2.

Parallelizability:

We call a mfld *parallelizable* if its tangent bdl is trivial; equivalently M^n is parallelizable iff it admits n vector fields (sections of $\begin{smallmatrix} TM \\ \downarrow \\ M \end{smallmatrix}$) which are everywhere linearly independent.

Thm (Stiefel, MS 4.6): If $\mathbb{R}P^n$ is parallelizable, then $n+1 = 2^r$ for some r.

In fact, only $\mathbb{R}P^1$, $\mathbb{R}P^3$ and $\mathbb{R}P^7$ are parallelizable, but that's harder.

<u>Lemma [MS 4.4]</u> Let X be a p.cpt Hausdorff space.

If $\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$ is an $\mathbb{R}^n$-bdle with k linearly independent sections, then $w_n(E) = \cdots = w_{n-k+1}(E) = 0$.

Similarly, if $\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$ is a $\mathbb{C}^n$-bdle w/ k lin. indep. sections, then $c_n E = \cdots = c_{n-k+1} E = 0$.

<u>Note</u>: A collection of sections is called <u>lin. indep</u> if its values at <u>every point</u> in X are lin. indep. The weaker statement that sections are some<u>where</u> lin. indep. is not very useful.

<u>Proof</u>: The k sections span a subbundle of dim'n k, which is trivial. We can put a metric on E, and then write

$$E \cong \underbrace{\varepsilon^k}_{\text{trivial } k\text{-plane bdle.}} \oplus (\varepsilon^k)^\perp$$

So $w(E) = w(\varepsilon^k)^\perp$, and $(\varepsilon^k)^\perp$ is an $(n-k)$-plane bdle, so its char. classes vanish above dim'n $n-k$. $\square$

In particular, if M^n is parallelizable, then $w(M^n) = 1$, which is more obvious (b/c TM^n is then trivial).

Proof of Thm 4.6:

We want to show that if $w(\mathbb{R}P^n) = 1$, then $n = 2^r - 1$ for some r. In fact, $w(\mathbb{R}P^n) = 1 \Leftrightarrow n = 2^r - 1$:

$\Leftarrow$ If $n = 2^r - 1$, then

$$w(\mathbb{R}P^n) = (1+\alpha)^{n+1} = (1+\alpha)^{2^r} = 1 + \alpha^{2^r} = \underbrace{1 + \alpha^{n+1}}_{=1},$$

b/c $\alpha^{n+1} \in H^{n+1}(\mathbb{R}P^n) = 0$ automatically vanishes.

$\Rightarrow$ We'll show that if $n \neq 2^r - 1$, then $w(\mathbb{R}P^n) \neq 1$.

Since $n+1 \neq 2^r$, we can write $n+1 = 2^r m$ for some odd integer $m > 1$. We have

$$(1+\alpha)^{n+1} = \left((1+\alpha)^{2^r}\right)^m = \left(1 + \alpha^{2^r}\right)^m$$
$$= 1 + m\alpha^{2^r} + \binom{m}{2} \alpha^{2^{r+1}} + \cdots .$$

But m is odd, and $2^r < n+1$, so $m\alpha^{2^r} = \alpha^{2^r} \in H^{2^r}(\mathbb{R}P^n; \mathbb{Z}/2)$ is non-zero. $\square$