**Lecture 11:** We must show that for any $x_k \in H^{*-1}B'$, 

$$\Phi^{rel}\partial\left(\sum_k x_k \otimes \alpha^k\right) = \partial\Phi'\left(\sum_k x_k \otimes \alpha^k\right).$$

We have

$$\Phi^{rel}\partial\left(\sum x_k \otimes \alpha^k\right) = \Phi^{rel}\left(\sum (\partial x_k) \otimes \alpha^k\right) = \sum q^*(\partial x_k) \cup (c_1 L_E)^k$$
$$= \sum \partial q^* x_k \cup (c_1 L_E)^k$$

and

$$\partial\Phi'\left(\sum x_k \otimes \alpha^k\right) = \partial\left(\sum (q^* x_k) \cup i^*(c_1 L_E)^k\right) = \sum \partial(q^* x_k \cup i^*(c_1 L_E)^k).$$

So we need to show that $\partial(q^* x_k \cup i^*(c_1 L_E)^k) = \partial(q^* x_k) \cup (c_1 L_E)^k$.

**Lemma:** For any pair $(X, A)$, the bdry map in the LES

$$H^{*-1}X \xrightarrow{i^*} H^{*-1}A \xrightarrow{\partial} H^*(X,A) \longrightarrow H^*X$$

satisfies $\partial(\alpha \cup i^* x) = \partial\alpha \cup x$.

**Proof:** The map $\partial$ is defined as follows: if $\tilde{a} \in C^{*-1}X$ is any cochain restricting to a cocycle $a \in C^{*-1}A$, then $\partial([a]) := [\delta(\tilde{a})]$, which is a cochain vanishing on $C_*A$, i.e. $\delta(\tilde{a}) \in C^*(X,A) \subseteq C^*X$.

Now, given $x = [X] \in H^{*-1}X$ and $\alpha = [a] \in H^{*-1}A$, choose $\tilde{a} \in C^{*-1}X$ extending $a$. Then $\tilde{a} \cup x \in C^{2*-2}X$ restricts to $a \cup i^* x$, so

$$\partial(\alpha \cup i^* x) = \partial[a \cup i^* x] = \delta(\tilde{\alpha} \cup x)$$
$$\underset{\boxed{\delta \text{ is a derivation}}}{=} (\delta\tilde{\alpha}) \cup x \pm \tilde{\alpha} \cup \delta x \underset{\boxed{X \text{ is a cocycle so } \delta x = 0}}{=} \delta\tilde{\alpha} \cup x. \qquad \square$$

By the 5-lemma, to complete the argument for fin. dim'l CW base spaces, it suffices to prove:

Claim 1: $\Phi'$ is an isomorphism.

Claim 2: $\Phi^{rel}$ is an isomorphism

Proof of Claim 1: We have a commutative diagram of fibrations

$$\begin{tikzcd} \mathbb{C}P^{n-1} \arrow[r, equal] \arrow[d] & \mathbb{C}P^{n-1} \arrow[d] \\ q^{-1}(B^{(n-1)}) \arrow[r, hook] \arrow[d, "q"] & q^{-1}(B') \arrow[d, "q"] \\ B^{(n-1)} \arrow[r, hook] & B' \end{tikzcd}$$

. Since $D^n - \{pt\}$ deformation retracts to $S^{n-1}$, $B'$ deformation retracts to $B^{(n-1)}$.

The 5-lemma now implies that $q^{-1}(B^{(n-1)}) \hookrightarrow q^{-1}(B')$ induces an isomorphism on $\pi_*$, and hence on $H^*$ (by Hatcher Prop. 4.21)

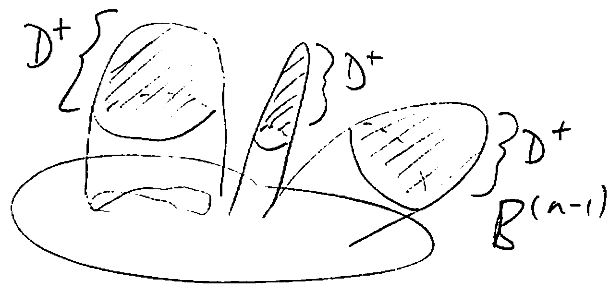
The diagram

$$\begin{tikzcd} H^* q^{-1}(B^{(n-1)}) \arrow[r, "\cong"] & H^*(q^{-1}B') \\ H^*B^{(n-1)} \otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[r, "\cong"] \arrow[u, "\Phi^{(n-1)}"] & H^*B' \otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[u, "\Phi'"] \end{tikzcd}$$

completes the proof of Claim 1, since by induction $\Phi^{n-1}$ is an isomorphism. $\square$

## Proof of Claim 2:

Let $D^+ \subseteq B$ denote the disjoint union of the closed, radius $\frac{1}{2}$ disks inside each $n$-cell in $B$:




$D^+ =$ shaded region

Then $D^+$ deformation retracts to the disjoint union of points $B - B'$, and $D^+ \cap B'$ deformation retracts to the disjoint union of the boundary $(n-1)$-spheres of the various disks in $D^+$. Note that these are both CW cplxs of dim'n $< n$, so the theorem applies to them by induction.

We have excision isomorphisms $H^*(B, B') \cong H^*(D^+, D^+ \cap B')$ and $H^*(PE, PE') \cong H^*(PE|_{D^+}, PE|_{D^+ \cap B'})$, coming from excising the complements of $D^+$ and of $PE|_{D^+}$ respectively. This gives a comm. diagram

$$\begin{tikzcd}
H^*(PE, PE') \arrow[r, "\cong"] & H^*(PE|_{D^+}, PE|_{D^+ \cap B'}) \\
H^*(B, B') \otimes H^*\mathbb{C}P^{n-1} \arrow[u] \arrow[r, "\cong"] & H^*(D^+, D^+ \cap B') \otimes H^*\mathbb{C}P^{n} \arrow[u]
\end{tikzcd}$$

and hence it suffices to prove that the right hand map is an isom.

But this follows by applying the induction hypothesis to the LES's of the pairs $(PE|_{D^+}, PE|_{D^+\cap B'})$, $(D^+, D^+\cap B')$: we have a diagram

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^{*-1}(PE|_{D^+\cap B}) \arrow[r, "\partial"] & H^*(PE|_{D^+}, PE|_{D^+\cap B'}) \arrow[r] & H^*PE|_{D^+} \arrow[r] & H^*PE|_{D^+\cap B'} \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{*-1}(D^+\cap B)\otimes H^*\mathbb{C}P^{n-1} \arrow[r, "\partial"] \arrow[u] & H^*(PE|_{D^+}, PE|_{D^+\cap B'})\otimes H^*\mathbb{C}P^{n-1} \arrow[r] \arrow[u, "\Phi^{rel}"] & H^*PE|_{D^+}\otimes H^*\mathbb{C}P^{n-1} \arrow[r] \arrow[u] & H^*(D^+\cap B')\otimes H^*\mathbb{C}P^{n-1} \arrow[u] &
\end{tikzcd}$$

and the argument in the proof of Claim 1 shows that the vertical arrows other than $\Phi^{rel}$ are isom's. The 5-Lemma completes the proof. $\square$

We have now shown that the Proj. Bdle Thm holds for any finite dim'l CW cplx.

If $B$ is a possibly infinite-dim'l CW complex, then we need to show that the map

$$\Phi : H^*B\otimes H^*\mathbb{C}P^{n-1} \stackrel{\text{def}}{=} \bigoplus_{\substack{*-k \text{ even} \\ *-k>0}} H^kB\otimes \underbrace{H^{*-k}\mathbb{C}P^n}_{\mathbb{Z}} \longrightarrow H^*PE$$

is an isomorphism. If we choose $N > *+1$ then by the cellular approximation theorem we have $\pi_i B^{(N)} \xrightarrow{\cong} \pi_i B$ for $i \leq N-1$, and in particular for $i \leq *+1$. Hence by [Hatcher 4.21], $B^{(N)} \hookrightarrow B$ (and consequently, $PE|_{B^{(N)}} \hookrightarrow PE$) induce isom's on $H^k$ for $k \leq *$. The result now follows from the diagram

$$\begin{array}{ccc} H^* PE|_{B^{(n)}} & \xleftarrow{\cong} & H^* PE \\ \uparrow \cong & & \uparrow \\ H^* B^{(n)} \otimes H^* \mathbb{C}P^{n-1} & \xleftarrow{\cong} & H^* B \otimes H^* \mathbb{C}P^{n-1} \end{array}$$

.

Finally, consider an arbitrary paracompact base space $B$. Then as shown in Hatcher Prop'n 4.13 there exists a "CW approximation" $K \xrightarrow{f} B$, i.e. $K$ is a CW cplx and $f_* : \pi_* K \to \pi_* B$ is an isom. for all $*$ (hence $H^* B \to H^* K$ is also an isom.).

The diagram

$$\begin{array}{ccc} H^* PE & \xrightarrow{\cong} & H^*(P(f^* E)) \\ \uparrow & & \uparrow \cong \\ H^* B \otimes H^* \mathbb{C}P^n & \xrightarrow{\cong} & H^* K \otimes \mathbb{C}P^n \end{array}$$

now completes the proof; the top map comes from the isomorphism $P(f^* E) \cong f^* P(E)$ and the diagram

$$\begin{array}{ccc} f^* PE & \longrightarrow & PE \\ \downarrow & & \downarrow \\ K & \xrightarrow{f} & B \end{array}$$

(the top map is an isom on $\pi_*$ and hence $H^*$ by the 5-lemma). $\square$

Rmk: Where did we use paracompactness of the base space $B$??. For **this argument to make** sense, we need to know that the class $c_1(L_E) \in H^*PE$ is defined. This means the line bdle $\begin{matrix} L_E \\ \downarrow \\ PE \end{matrix}$ must admit a classifying map
$$\begin{tikzcd} L_E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \gamma_1 \arrow[d] \\ PE \arrow[r, "f"] & \mathbb{C}P^\infty \end{tikzcd}$$
, which we can only guarantee if $PE$ is paracompact [MS §5].

On the other hand, in the real case we saw that $w_1\left(\begin{matrix} L \\ \downarrow \\ X \end{matrix}\right) \in H^1(X;\mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1 X, \mathbb{Z}/2)$ could be defined over any base space. So we don't need paracptness when the bundle is a real vector bdle.