Lecture 10

Cohomology of Projective Spaces

Before addressing the Projective Bdle Thm, we consider the case of computing $H^*(\mathbb{R}P^n;\mathbb{Z}/2)$ and $H^*(\mathbb{C}P^n;\mathbb{Z})$.

Prop'n: For each n, $\mathbb{R}P^n$ is a CW cplx with one e_k in each dimension $1 \le k \le n$. The $(n-1)$-skeleton of $\mathbb{R}P^n$ is precisely $\mathbb{R}P^{n-1}$, and the attaching map $\psi_n : S^{n-1} \to \mathbb{R}P^{n-1} = (\mathbb{R}P^n)^{(n-1)}$ is just the quotient map defining $\mathbb{R}P^{n-1}$.

Pf: By induction on n. For $\mathbb{R}P^1 = S^1$, there is nothing to prove. Now, $\mathbb{R}P^n = S^n/_{x\sim -x}$. If $D^n \hookrightarrow S^n$ is inclusion of the upper hemisphere, then the induced map $D^n/_{\substack{x\sim -x \\ \text{for } x\in\partial D^n}} \to S^n/_{x\sim -x} = \mathbb{R}P^n$ is a homeomorphism. But the subspace $\partial D^n/_{x\sim -x} \subseteq D^n/_{\substack{x\sim -x \\ \text{for } x\in\partial D^n}}$ is precisely $\mathbb{R}P^{n-1}$! So $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_{\partial D^n} D^n$ and the attaching map is as described. $\square$

Corollary 1: $H^k(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for $0 \le k \le n$, and is 0 otherwise.

Proof: Since $H^k(X;\mathbb{Z}/2) \cong \mathrm{Hom}(H_k(X;\mathbb{Z}/2), \mathbb{Z}/2)$ (Univ. Coeff. Thm.) it suffices to compute $H_k(\mathbb{R}P^n;\mathbb{Z}/2)$. We will compute

$H_*(\mathbb{R}P^n; \mathbb{Z}/2)$ via cellular homology. The cellular chain cplx has the form

$$0 \to \underbrace{\mathbb{Z}/2 \xrightarrow{d_n} \mathbb{Z}/2 \xrightarrow{d_{n-1}} \mathbb{Z}/2 \to \cdots \xrightarrow{d_2} \mathbb{Z}/2 \xrightarrow{d_1} \mathbb{Z}/2}_{n+1} \to 0,$$

where d_k is the cellular bdry map. It is simply mult'n by the degree (mod 2) of the composite

$$S^{k-1} \xrightarrow{\psi_k} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \underset{\varphi_{k-1}^{-1}}{\xrightarrow{\cong}} D^{k-1}/\partial D^{k-1} = S^{k-1}$$

where $\varphi_{k-1} : D^{k-1} \to \mathbb{R}P^n$ is the characteristic map for the $(k-1)$-cell. This map restricts to homeomorphisms $S^{k-1}_+ \xrightarrow{\cong} D^{k-1} - \partial D^{k-1}$ and $S^{k-1}_- \xrightarrow{\cong} D^{k-1} - \partial D^{k-1}$ ($S^{k-1}_\pm$ the hemispheres) so the degree is the (signed) number of inverse images of any $x \in D^{k-1} - \partial D^{k-1}$, which is $(\pm 1) + (\pm 1) = 0$ or $\pm 2 = 0 \pmod 2$. So all the d_k are 0, and we have $H_k(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2$ for $0 \le k \le n$, and $H^k(\mathbb{R}P^n; \mathbb{Z}/2) = \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ for $0 \le k \le n$. $\square$

Before computing the ring structure, we consider the complex case.

<u>Prop'n</u>: $\mathbb{C}P^n$ is a CW complex with one cell in each even dimension $(0, 2, 4, \ldots, 2n)$. The $(2n-2)$-skeleton of $\mathbb{C}P^n$ is $\mathbb{C}P^{n-1}$, and the $2n$ cell is attached via $\partial D^{2n} = S^{2n-1} \xrightarrow{q} \mathbb{C}P^{n-1}$, the quotient map.

<u>Proof</u>: $\mathbb{C}P^n = S^{2n+1} / \underset{\text{for } \lambda\in S^1\subset\mathbb{C}}{v\sim\lambda v}$, and we need to view this as $D^{2n} / \underset{\substack{\text{for } v\in\partial D^{2n}=S^{2n-1}\\ \lambda\in S^1\subset\mathbb{C}}}{v\sim\lambda v}$; the latter is precisely a $2n$-cell attached to $\mathbb{C}P^{n-1} = S^{2n-1}/_{v\sim\lambda v}$ via the quotient map. The disk we will use is $D^{2n}_+ = \{(z_1,\dots,z_{n+1}) \in S^{2n+1}\subset\mathbb{C}^{n+1} \mid z_{n+1}\in\mathbb{R}_{\geq 0}\}$, which is just a hemisphere. The map $D^{2n}_+ \longrightarrow S^{2n+1} \xrightarrow{q} \mathbb{C}P^n$ is onto, and it's injective on $\{(z_1,\dots,z_{n+1})\in D^{2n}_+ \mid z_{n+1}>0\}$ (b/c under rotation by $\lambda = e^{i\theta}\in S^1$, each $z\in\mathbb{C}-\{0\}$ gets identified with exactly one pt. on $\mathbb{R}_{>0}$). So $D^{2n}_+ / \underset{\text{for } v\in\partial D^{2n}}{v\sim\lambda v} \cong \mathbb{C}P^n$. $\square$

<u>Corollary</u>: $H_*(\mathbb{C}P^\infty;\mathbb{Z}) = H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \begin{cases}\mathbb{Z}, & * \text{ even}\\ 0, & * \text{ odd}\end{cases}$.

<u>Pf</u>: The cellular chain cplx has the form

$$0\to\mathbb{Z}\to 0\to\mathbb{Z}\to 0\to\cdots\to\mathbb{Z}\to 0\to\mathbb{Z}\to 0$$

b/c all the cells are in even dim'ns. So the bdry maps are automatically zero. The same goes for cohomology. $\square$

The cup product structure on $H^* \mathbb{R}P^n$, $H^* \mathbb{C}P^n$ can be computed via somewhat delicate geometric arguments (see Hatcher) but can also be deduced from Poincaré Duality.

Prop (Hatcher 3.38): The cup product pairing

$$(\star) \qquad H^k(M;\mathbb{Z}) \times H^{n-k}(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$\varphi \quad , \quad \psi \longmapsto (\varphi \cup \psi)([M])$$

is non-singular for any orientable mfld M with $H^*(M;\mathbb{Z})$ torsion-free.

Similarly, the corresponding pairing with $\mathbb{Z}/2$-coeffs is non-singular **for any** mfld.

Rmk: Such a bilinear pairing $A \times B \longrightarrow R$ is called non-singular if the induced maps

$$A \longrightarrow \mathrm{Hom}(B,R) \quad , \quad B \longrightarrow \mathrm{Hom}(A,R)$$

are isomorphisms.

Proof: We can factor the map $H^k \to \mathrm{Hom}(H^{n-k}, \mathbb{Z})$
$$\varphi \longmapsto (\psi \mapsto (\varphi \cup \psi)[M])$$

as follows:

$$H^k(M;\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_{n-k}(M,\mathbb{Z}), \mathbb{Z}) \longrightarrow \mathrm{Hom}(H^{n-k}(M,\mathbb{Z}),\mathbb{Z}),$$
$$\varphi \longmapsto (\sigma \mapsto \varphi(\sigma)) \qquad f \longmapsto (\psi \mapsto f(\psi \cap [M]))$$

b/c the composite is $\varphi \mapsto \left(\psi \mapsto \varphi(\psi \cap [M]) \overset{\text{check at chain level}}{=} (\varphi \cup \psi)[M]\right)$. The first map is an isom. by the UCT, the 2nd by Poincaré Duality. The $\mathbb{Z}/2$ case is the same. Switching k and $n-k$ completes the proof. $\square$

<u>Corollary</u>: • $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, with $|\alpha|=2$

$\boxed{\text{ring isomorphisms}}$

• $H^*(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1})$, with $|\alpha|=1$.

<u>Pf</u>: There is a ring map $\mathbb{Z}[\alpha]/(\alpha^{n+1}) \to H^*(\mathbb{C}P^n;\mathbb{Z})$, defined by sending $\alpha \mapsto \alpha = c_1(\gamma_1) \in H^2(\mathbb{C}P^n;\mathbb{Z})$ (the canonical generator). By our additive computation of $H^*(\mathbb{C}P^n;\mathbb{Z})$, it will suffice to show that $\alpha^k \in H^{2k}(\mathbb{C}P^n;\mathbb{Z})$ is a generator (for $k=2,\dots,n$). We prove this by induction on n. For $n=1$, there is nothing more to prove, since we know α generates $H^2(\mathbb{C}P^1;\mathbb{Z})$. The inclusions $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induce isomorphisms $H^k\mathbb{C}P^{n-1} \xleftarrow{\cong} H^k\mathbb{C}P^n$ for $k \le 2n-2$, since $\mathbb{C}P^n$ is formed from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell. (In the real case, $H^k(\mathbb{R}P^{n-1};\mathbb{Z}/2) \xleftarrow{\cong} H^k(\mathbb{R}P^n;\mathbb{Z}/2)$ for $k \le n-1$, because the cellular boundary maps in $C^*(\mathbb{R}P^n;\mathbb{Z}/2)$ are all zero.) So by induction we may assume that α^k generates $H^{2k}(\mathbb{C}P^n;\mathbb{Z})$ for $k<n$.

By the Prop'n, the homomorphism $H^2(\mathbb{C}P^n;\mathbb{Z}) \xrightarrow{f} \mathbb{Z}$, $\alpha \longmapsto 1$

is given by $\psi \longmapsto (\varphi \cup \psi\,[\mathbb{C}P^n])$ for some class $\varphi \in H^{2n-2}(\mathbb{C}P^n;\mathbb{Z})$. So

$$1 = f(\alpha) = \varphi \cup \alpha\,[\mathbb{C}P^n],$$

which means that neither φ nor $\varphi \cup \alpha$ can be written as $m\beta$ with $m>1$. Then φ generates $H^{2n-2}(\mathbb{C}P^n;\mathbb{Z})$, and we have $\varphi = \pm\alpha^{n-1}$ by induction.

Now $\varphi \cup \alpha = \pm\alpha^n$, and again this element must generate $H^{2n}(\mathbb{C}P^n;\mathbb{Z})$ (otherwise $\varphi \cup \alpha\,[\mathbb{C}P^n] = \pm\alpha^n[\mathbb{C}P^n]$ could not be 1). So α^n generates $H^{2n}(\mathbb{C}P^n;\mathbb{Z})$, as desired.

$\square$

Proof of the Projective Bdle Theorem:

First we recall the statement.

Theorem: Let $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ be a cplx vector bdle, with B paracpt, and let $\begin{matrix} P(E) \\ q\downarrow \\ B \end{matrix}$ be the corresponding projective space bdle.

Then the map $q^*: H^*(B;\mathbb{Z}) \to H^*(PE;\mathbb{Z})$ is injective, and the induced map

$$H^*(B) \otimes_{\mathbb{Z}} \underbrace{\mathbb{Z}[a, a^2, \ldots, a^k]}_{\substack{\text{graded free abelian} \\ \text{group on } k \text{ generators,} \\ |\alpha^i| = 2i}} \longrightarrow H^*(P(E);\mathbb{Z})$$

sending $a^i \longrightarrow c_1(L_E)^i$ is an isomorphism of H^*B-modules.
The same result holds for real bdles, with $\mathbb{Z}$ replaced by $\mathbb{Z}/2$.

Proof: We begin by proving the theorem for CW cplxs B, starting with finite dim'l complexes. If B is zero-dim'l, then $P(E) = \coprod_{b \in B} \mathbb{C}P^n$, so the result follows from our computation of $H^*(\mathbb{C}P^n;\mathbb{Z})$, and the fact that the restriction of L_E to each copy of $\mathbb{C}P^n$ is the tautological bdle $\begin{matrix} \gamma_1^n \\ \downarrow \\ \mathbb{C}P^n \end{matrix}$.

Now assume the result for $(n-1)$-dim'l complexes, and say B is n-dim'l. We will consider the following diagram, in which $B' \subseteq B$ denotes the subspace formed by removing one point from the interior of each n-cell, and we replace the polynomial ring by $H^*\mathbb{C}P^n$:

$$(\star)\quad \begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^{*-1}(q^{-1}B') \arrow[r, "\partial"] & H^*(PE, q^{-1}B') \arrow[r, "j^*"] & H^*(PE) \arrow[r, "i^*"] & H^*(q^{-1}B') \arrow[r] & \cdots \\
& H^{*-1}(B')\otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[r, "\partial\otimes id"] \arrow[u, "\Phi'"] & H^*(B,B')\otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[r, "j^*\otimes id"] \arrow[u, "\Phi^{rel}"] & H^*(B)\otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[r, "i^*\otimes id"] \arrow[u, "\Phi"] & H^*(B')\otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[r] \arrow[u, "\Phi'"] & \cdots
\end{tikzcd}$$

The top row of $(\star)$ is just the LES of the pair $(PE, q^{-1}B')$. The bottom row requires some explanation. We use $H^* q^{-1}B' \otimes_{\mathbb{Z}} H^*\mathbb{C}P^n$ as short-hand for the $*$-th graded piece of the graded tensor product; this means that the lower LES is actually the sum of various shifted copies of the LES for the pair (B,B'); here we use that $H^k(\mathbb{C}P^{n-1})$ is always either $\mathbb{Z}$ or 0. The vertical maps Φ, Φ' are induced by the map of pairs

$$(PE, q^{-1}B') \xrightarrow{q} (B,B')$$

and the assignments $\alpha^k \mapsto c_1(L_E)^k$ or $\alpha^k \mapsto i^*(c_1 L_E)$ ($\alpha \in H^2\mathbb{C}P^n$ is the canonical generator). The map Φ^{rel} is defined similarly, via the relative cup product:

$$H^*(B,B')\otimes H^*\mathbb{C}P^{n-1} \xrightarrow{q^*\otimes(\alpha\mapsto c_1 L_E)} H^*(PE, q^{-1}B')\otimes H^* PE \xrightarrow{\cup} H^*(PE, q^{-1}B').$$

Commutativity of the middle and right squares is immediate, and we must check commutativity of the left square. We must show that for any $x_i \in H^{*-1}B'$,