Math 372B

HW 4

Please turn in sol'ns to all 3 problems by Friday, 4/24.

1. Recall that the exterior powers of a vector space W are defined as follows:

$$T(W) = \bigoplus_{k=0}^{\infty} \underbrace{W\otimes\cdots\otimes W}_{k} \quad (\text{When } k=0,\ \underbrace{W\otimes\cdots\otimes W}_{k} \overset{\text{def}}{=} \mathbb{R})$$

is the tensor algebra, and the exterior algebra is

$$\Lambda(W) = T(W) \Big/ \left(\{w\otimes w \mid w\in W\}\right)$$

2-sided ideal I wrt mult'n $(w_1\otimes\cdots\otimes w_k)(w_{k+1}\otimes\cdots\otimes w_\ell) = w_1\otimes\cdots\otimes w_\ell$

We write elts in $\Lambda(W)$ as $w_1\wedge\cdots\wedge w_k = w_1\otimes\cdots\otimes w_k + I$.

The k^{th} exterior power of W is

$$\Lambda^k W = \{w_1\wedge\cdots\wedge w_k \in \Lambda W\}.$$

a) Show that $\Lambda^k(-)$ is a functor $\underline{Vect}_{\mathbb{R}} \longrightarrow \underline{Vect}_{\mathbb{R}}$, so that $\Lambda^k(\xi)$ is defined for any bdle ξ, and prove that if $\dim\xi = k$, then $\Lambda^k\xi$ is a line bdle.

b) Prove that ξ^k ← k-dim'l is orientable $\Longleftrightarrow$ $\Lambda^k\xi^k$ is trivial, and conclude that $w_1\xi^k = w_1\Lambda^k\xi^k$.

c) Describe the clutching fcns of $\Lambda^k(\xi^k)$ in terms of the clutching fcns for ξ^k.

Rmk: Since $\begin{matrix}\Lambda^k(\xi^k)\\ \downarrow\\ B\end{matrix}$ is a line bdle, it's classified by a map $B\longrightarrow \mathbb{RP}^\infty = B\mathbb{Z}/2$. If B is a CW cplx, then $[B, B\mathbb{Z}/2] \cong H^1(B;\mathbb{Z}/2)$, $f \longmapsto f^*(w_1)$, so ξ^k orientable $\Leftrightarrow$ $\Lambda^k\xi^k$ trivial $\Leftrightarrow$ $w_1\Lambda^k\xi^k = 0 \Longleftrightarrow w_1\xi^k = 0$.

2. In this problem, we'll construct the Mayer-Vietoris sequence in K-theory.

Let $X = A \cup B$, with X a finite CW cplx and A, B subcomplexes. Consider the square

$$\begin{array}{ccc} \mathrm{Map}_*(X, BU) & \longrightarrow & \mathrm{Map}_*(A, BU) \\ \downarrow & & \downarrow \\ \mathrm{Map}_*(B, BU) & \longrightarrow & \mathrm{Map}_*(A \cap B, BU). \end{array}$$

a) Show that the vertical maps are fibrations with the same fiber (use HW 2)

b) Combine the two resulting LES's to obtain a M-V sequence

$$\xrightarrow{\partial} \pi_k \mathrm{Map}_*(X, BU) \xrightarrow{(i_A^*, -i_B^*)} \pi_k \mathrm{Map}_*(A, BU) \oplus \pi_k \mathrm{Map}_*(B, BU)$$

$$\longrightarrow \pi_k \mathrm{Map}_*(A \cap B, BU) \xrightarrow{\partial} \pi_{k-1} \mathrm{Map}_*(X, BU)$$

(this is Hatcher section 2.2 Problem 38).

c) Show that $\pi_{2k} \mathrm{Map}_*(X, BU) = \tilde{K}^0(X)$

$\pi_{2k+1} \mathrm{Map}_*(X, BU) = K^1(X)$

(use Bott Periodicity).

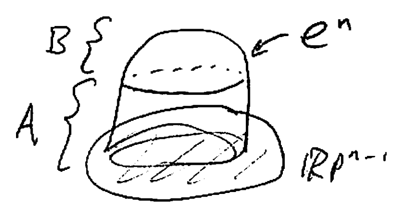
Remark: Parts a), b) would work just fine with BU replaced by any other space, e.g. $BU(n)$. So in some sense, the Mayer-Vietoris sequence in K-theory is an "unstable" phenomenon.

3. Use the Mayer-Vietoris sequence from #2 to prove that for any $n \geq 1$,

- $\tilde{K}^0 \mathbb{RP}^{2n-1}$ is a finite group of order 2^{n-1}
- $K^1 \mathbb{RP}^{2n-1} \cong \mathbb{Z}$
- $\tilde{K}^0 \mathbb{RP}^{2n}$ is a finite group of order 2^n
- $K^1(\mathbb{RP}^{2n}) = 0.$

Hint: Use the M-V sequence arising from attaching the top dim'l cell to $\mathbb{RP}^n$:

$\underline{\mathbb{RP}^n}$:



You'll need to compute the effect of the quotient map

$$S^{2n-1} \longrightarrow \mathbb{RP}^{2n-1}$$

on K^1; you can do this using the Chern character and the computation of $H_*(\mathbb{RP}^{2n-1};\mathbb{Z})$ in Hatcher (p.144):

$$H_*(\mathbb{RP}^{2n-1};\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k=0 \text{ and } k=2n-1 \\ \mathbb{Z}/2 & \text{for } k \text{ odd}, \ 0<k<2n-1 \\ 0 & \text{else.} \end{cases}$$