Math 372B, HW3

Please turn in at least 3 solutions by Thursday, 3/26

1. a) Prove the following alternate form of the Splitting Principle": For any $n \geq 1$, the map

$$\underbrace{Gr_1(\mathbb{C}^\infty) \times \cdots \times Gr_1(\mathbb{C}^\infty)}_{n} \xrightarrow{\alpha} Gr_n(\mathbb{C}^\infty) \quad (\star)$$

classifying the complex n-plane bundle $\overbrace{\gamma_1 \times \cdots \times \gamma_1}^{n}$ induces an injection on integral cohomology. (Hint: use the Projective Bundle Theorem.)

b) Deduce that there can be no algebraic relations amongst the elementary symmetric polynomials. (Think about $\alpha^*(c_i(\gamma_n)) \in H^*((Gr_1\mathbb{C}^\infty)^n;\mathbb{Z})$, where α is the map in $(\star)$.)

c) Deduce that if $\sum_{k=1}^{n} n_i = n$, then the classifying map for $\gamma_{n_1} \times \cdots \times \gamma_{n_k}$ induces an injection on $H^*(-;\mathbb{Z})$.

2. (Some computations)

a) Show that the Stiefel-Whitney classes of the tangent bundle $T(S^n)$ are all trivial. (Hint: consider the normal bundle, i.e. the orthogonal complement of $TS^n \subseteq T\mathbb{R}^{n+1}$.)

b) Show that if M is an orientable manifold, then $w_1(TM) = 0$. (Hint: use our alternative def'n of w_1 involving loops.)

c) Show that if M is a 2-dim'l orientable mfld (so $M = M_g$ = $\overbrace{\text{(surface)}}^{g\text{-holes}}$) then $w_1 TM^g = w_2 TM^g = 0$.

3. Let E and F be complex 2-plane bdles over X.

Compute $c_i(E\otimes F)$ via the Splitting Principle (your answer should be some formula in terms of $c_i E$ and $c_j F$.

2 Problems involving the Chern Character:

4. Show that if M^{2i+1} is an odd dimensional, closed, orientable mfld, the groups $K^0(M)$ and $K^1(M)$ have the same rank.

5. Show that if $H^{2i-1}(X;\mathbb{Q}) \neq 0$, then there exists a cplx vector bundle $\begin{matrix} E \\ \downarrow \\ \Sigma X \end{matrix}$ with $c_i(E)\neq 0$ in $H^{2i}(\Sigma X;\mathbb{Z}) \cong H^{2i-1}(X;\mathbb{Z})$.

6. Prove that if $f: X \longrightarrow Y$ is a map b/w finite CW cplxs which induces an injection

$$H^*(Y;\mathbb{Z}) \xhookrightarrow{f^*_{\mathbb{Z}}} H^*(X;\mathbb{Z})$$

then it induces an injection

$$H^*(Y;\mathbb{Q}) \xhookrightarrow{f^*_{\mathbb{Q}}} H^*(X;\mathbb{Q}).$$

(Hint: show that $f^*_{\mathbb{Q}}$ can be identified (naturally) with

$$H^*(Y;\mathbb{Z})\otimes\mathbb{Q} \xrightarrow{f^*_{\mathbb{Z}}\otimes \mathrm{id}_{\mathbb{Q}}} H^*(X;\mathbb{Z})\otimes\mathbb{Q};$$

the result then follows from the fact that $\mathbb{Q}$ is a "flat" $\mathbb{Z}$-module.

Hint: The Univ. Coeff Thm for chain complexes (of finitely gen'd free ab gps) works equally well for cochain complexes of such gps.