Math 372B, HW2

1. Let $\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & X & \end{array}$ be a map of vector bundles, and assume that $\mathrm{Rank}_x(f) = \dim\left(f(\pi_1^{-1}(x))\right) \subseteq \pi_2^{-1}(x)$ is constant. Prove that $\mathrm{Im}(f)$ and $\mathrm{Ker}(f)$ are (locally trivial) vector bundles over X. (The image and kernel are simply the collections of all vectors in $\mathrm{im}(f_x : \pi_1^{-1}(x) \to \pi_2^{-1}(x))$ or $\ker(f_x : \pi_1^{-1}(x) \to \pi_2^{-1}(x))$.)

[Hint: this is similar to the proof, in MS, that the orthogonal complement of a subbundle is locally trivial (MS Thm 3.3), which we used in class to write the pullback $\begin{array}{ccc} q^*E & \to & E \\ \downarrow & & \downarrow \\ P(E) & \xrightarrow{q} & X \end{array}$ as a sum: $q^*E = L_E \oplus L_E^{\perp}$. It's also similar to the proofs, in the notes, that tangent bdles and Stiefel bdles are locally trivial.]

2. a) Let $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ be a vector bdle with transition fcns $\varphi_{ji} : U_i \cap U_j \to GL_n\mathbb{R}$ (or $GL_n\mathbb{C}$), where $\{U_i\}_i$ is an open cover of X over which E is trivial. Calculate the clutching fcns of the dual bdle $E^* = \mathrm{Hom}(E, \mathbb{R})$ over the same sets $U_i \cap U_j$.

b) Let $\begin{tikzcd}[column sep=small] E_1 \arrow[dr] & & E_2 \arrow[dl] \\ & X & \end{tikzcd}$ be as in a), and let $\varphi_{ji} : U_i \cap U_j \to GL_n\mathbb{R}$ and $\psi_{ji} : U_i \cap U_j \to GL_m\mathbb{R}$ be the transition fcns. Describe the transition fcns of $E_1 \otimes E_2$ in terms of φ_{ji} and ψ_{ji}.

3. (MS Problem 4-A) Compute the Stiefel-Whitney (or Chern) Classes of a Cartesian product $\begin{matrix} E \times F \\ \downarrow \\ X \times Y \end{matrix}$ in terms of the classes of $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ and $\begin{matrix} F \\ \downarrow \\ Y \end{matrix}$. (Hint: Find a way to apply the Whitney Sum Formula)

4. We say that bdles $\begin{tikzcd}[column sep=small] E \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$ are <u>stably</u> <u>isomorphic</u> if there exists $n \geq 0$ s.t. $E \oplus (X \times \mathbb{R}^n) \cong F$.

How are the Stiefel-Whitney Classes of E and F related if E is stably isomorphic to F? (The cplx case is identical.)

The (reduced) K-theory group $\tilde{K}_0(X)$ is the group $\mathrm{Vect}(X) / (\text{Stable isomorphism})$. Deduce that w_i, c_i are well defined <u>Functions</u> $\tilde{K}_0(X) \to H^*X$, where in the real case we use $\mathbb{Z}/2$-coeff's and in the cplx case we use $\mathbb{Z}$-coeff's.

5. a) Say $A \subseteq X$ is a CW pair (i.e. A is a subcomplex of X). Prove that for any space Y, the restriction function $\begin{array}{ccc} Map(X,Y) & & F \\ \downarrow & & \rotatebox[origin=c]{-90}{$\mapsto$} \\ Map(A,Y) & & F|_A \end{array}$ is a (Serre) Fibration. [Hint: as explained in Hatcher Chapter 0, all CW pairs are cofibrations, i.e. they satisfy the Homotopy Extension Property.]

Rmk: You'll need the following fact regarding the compact-open topology on mapping spaces: If W is locally compact and regular (e.g. a CW cplx) then a function $f : U \times W \to V$ is continuous if and only if its "adjoint" $U \to Map(W,V)$ is continuous. (The "only if" direction doesn't require any hypotheses on W.)

b) Show that the map $\begin{array}{c} P_{x_0}(X) \overset{def}{=} Map(([0,1],0),(X,x_0)), \\ \downarrow \\ X \end{array}$

defined by $\gamma \mapsto \gamma(1)$, is a fibration, whose fiber over $x_0 \in X$ is the based loop space $\Omega_{x_0}(X) = Map((S^1,1),(X,x_0))$.

c) Show that for any top. gp. G and any universal principal G-bdle $\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$ with EG contractible (not just $\pi_* EG = 0$), there is a weak htpy equivalence $\Omega BG \xrightarrow{f} G$, i.e. $f_* : \pi_* \Omega BG \to \pi_* G$ is an isom. [Hint: Compare the fibration $\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$ to the one in Part b).]

6. Show that the Gram-Schmidt orthogonalization process gives a <u>deformation</u> <u>retraction</u> $V_n(\mathbb{C}^{n+k}) \to V_n^0(\mathbb{C}^{n+k})$ (and similarly $V_n(\mathbb{R}^{n+k}) \to V_n^0(\mathbb{R}^{n+k})$). In other words, Check that the GS map is homotopic to Id_{V_n} through a htpy that fixes V_n^0 pointwise at all times.

Note that when $k=0$, this gives $GL_n\mathbb{C} \simeq U(n)$, $GL_n\mathbb{R} \simeq O(n)$.

7. a) Let $\varphi: G \to H$ be a continuous group homomorphism and let $\begin{matrix} EG \\ \downarrow p \\ BG \end{matrix}$, $\begin{matrix} EH \\ \downarrow q \\ BH \end{matrix}$ be universal principal bdles for G and for H. Show that there exists a diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & H \arrow[d] \\ EG \arrow[r, "\tilde{f}"] \arrow[d, "p"] & EH \arrow[d, "q"] \\ BG \arrow[r, "f"] & BH \end{tikzcd}$$

(with the maps $\begin{matrix} G \\ \downarrow \\ EG \end{matrix}$, $\begin{matrix} H \\ \downarrow \\ EH \end{matrix}$ equivariant isomorphisms onto fibers of p, q). You may assume that $\begin{matrix} EH \\ \downarrow \\ BH \end{matrix}$ classifies principal H-bdles over BG, which holds if 1) BG is a CW cplx; or 2) BG is paracompact and BH is either Milnor's model for the classifying space - with $EH = \ast_{1}^{\infty} H$, or $H = GL_n\mathbb{R}, O(n), GL_n\mathbb{C}, U(n)$ and $BH = Gr_n\mathbb{R}^\infty / Gr_n\mathbb{C}^\infty$.

b) Conclude that if φ is a weak htpy equiv. (e.g. see #6), then so is $BG \xrightarrow{f} BH$.

c) Conclude that the isomorphism $\pi_n BG \xrightarrow{\cong} \pi_{n-1} G$ (HW1, #3) is independent of EG so long as BG is a CW cplx.

d) Given $f: G \to H$, a principal G-bdle over X produces a principal H bdle over X by choosing clutching fcns $\{\varphi_{ij}\}$ for $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$, which are maps

$$\varphi_{ij} : U_i \cap U_j \to G$$

($\{U_i\}$ some cover of X), and forming an H-bdle via the clutching fcns

$$f \circ \varphi_{ij} : U_i \cap U_j \to H.$$

Using b), show that this process gives a well-defined bijection ~~between~~ b/w isom classes of G-bdles and isom. classes of H-bdles over any CW cplx X.

~~(Assume BG and BH are as in the earlier part of the problem~~

(Assume that nice models for BG and BH exist, as in the previous parts of the problem.)