Math 372B, Spring '09

Homework 1 : Due Thursday, 2/5/09

1. Show that the principal $GL_n\mathbb{R}$-bdle associated to a vector bdle $\begin{array}{c} V \\ \downarrow{\scriptstyle\pi} \\ B \end{array}$ is well-defined up to isomorphism. That is, show that if $\{(U_i, \varphi_i : U_i \times \mathbb{R}^n \to \pi^{-1}U_i)\}$ and $\{(V_j, \psi_j : V_i \times \mathbb{R}^n \to \pi^{-1}V_i)\}$ are two different trivializations of V, then the bdles $\left(\coprod_i U_i \times GL_n\mathbb{R}\right) \Big/ \begin{pmatrix} (u,A) \sim (u, \varphi_{ji}(u)A) \\ \text{for } u \in U_i,\ A \in GL_n\mathbb{R} \end{pmatrix}$ and $\left(\coprod_j V_j \times GL_n\mathbb{R}\right) \Big/ \begin{pmatrix} (v,A) \sim (v, \psi_{ji}(v)A) \\ \text{for } v \in V_j,\ A \in GL_n\mathbb{R} \end{pmatrix}$ are isomorphic.

2. a) Show that every map
$$\begin{tikzcd} P_1 \arrow[r, "\tilde{\varphi}"] \arrow[d] & P_2 \arrow[d] \\ B_1 \arrow[r, "\varphi"] & B_2 \end{tikzcd}$$
of principal $GL_n\mathbb{R}$-bundles induces a map
$$\begin{tikzcd} P_1 \times_{GL_n\mathbb{R}} \mathbb{R}^n \arrow[r, "\Phi"] \arrow[d] & P_2 \times_{GL_n\mathbb{R}} \mathbb{R}^n \arrow[d] \\ B_1 \arrow[r, "\varphi"] & B_2 \end{tikzcd}$$
of ("mixed") vector bundles.

b) Show that every map of vector bdles
$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[d] & W \arrow[d] \\ B_1 \arrow[r, "\varphi"] & B_2 \end{tikzcd}$$
induces a map of the associated principal bdles
$$\begin{tikzcd} P_V \arrow[r, "\tilde{\varphi}"] \arrow[d] & P_W \arrow[d] \\ B_1 \arrow[r, "\varphi"] & B_2 \end{tikzcd}.$$
Explain to what extent this map depends on a choice of trivialization.

3. Show that if $\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$ is a univ. princ. G-bdle, then there are canonical isomorphisms $\pi_* BG \cong \pi_{*-1} G$.

4. Given a free action of a top. gp. G on a space X, it is natural to ask whether the projection $\begin{array}{c} X \\ \downarrow q \\ X/G \end{array}$ is a principal G-bdle. Show that $\begin{array}{c} X \\ \downarrow q \\ X/G \end{array}$ is a principal G-bdle if and only if the action satisfies the following conditions:

1) $\forall\, x \in X$, $\exists$ a nbhd $U \subseteq X/G$ of $[x]$ and a section $\begin{array}{c} q^{-1}U \\ q\downarrow \uparrow s \\ U \end{array}$ (w/ $qs = id_U$).
(we say the action "admits slices").

2) The map $X \times_{X/G} X \longrightarrow G$, $(x, y) \longmapsto$ the unique $g \in G$ s.t. $x \cdot g = y$ is continuous.

5. (Uniqueness of Classifying Spaces) Say $\begin{array}{c} E_1 \\ \downarrow \\ B_1 \end{array}$ and $\begin{array}{c} E_2 \\ \downarrow \\ B_2 \end{array}$ are principal G-bdles with $\pi_* E_1 = \pi_* E_2 = 0$ for $* \geq 0$.

Show, using Problem 3., that B_1 and B_2 are weakly htpy equivalent: $\exists$ a space Z and maps $B_1 \xleftarrow{f} Z \xrightarrow{g} B_2$ s.t. f and g both induce isomorphisms on all htpy groups.

(Hint: Consider the action of G on $E_1 \times E_2$.)

6. Metrics on Vector Bdles:

a) (MS Problem 2C) Show, using a partition of unity, that every vector bdle over a paracompact base admits a metric. (Your proof should work for real and cplx bdles.)
Hint: the space of metrics is convex.

b) Using the Bundle Htpy Theorem, show if $\langle , \rangle_0$ and $\langle , \rangle_1$ are two metrics on the same vector bdle

$\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ (with X paracompact) then there is an isometry

$(V, \langle,\rangle_0) \xrightarrow{\cong} (V, \langle,\rangle_1)$.

Conclude that for X paracpt, there is a bijection b/w isomorphism classes of vector bdles over X and isometry classes of Euclidean (or Hermitian) bdles, and consequently bijections. $\mathrm{Prin}_{GL_n\mathbb{R}}(X) \cong \mathrm{Prin}_{O(n)}(X)$, $\mathrm{Prin}_{GL_n\mathbb{C}} X \cong \mathrm{Prin}_{U(n)}(X)$.

c) Let $BGL_n\mathbb{C}$ and $BU(n)$ be classifying spaces for $GL_n\mathbb{C}$ / $U(n)$ bdles (i.e. assume there are bdles $\begin{matrix} EGL_n\mathbb{C} \\ \downarrow \\ BGL_n\mathbb{C} \end{matrix}$, $\begin{matrix} EU(n) \\ \downarrow \\ BU(n) \end{matrix}$ s.t. the natural map $[X, BGL_n\mathbb{C}] \to \mathrm{Prin}_{GL_n\mathbb{C}} X$, $f \longmapsto f^* EGL_n\mathbb{C}$ and the corresponding one for $U(n)$ are bijections for all CW cplx's X).

By considering the clutching fcns for $EU(n)$ as elts of $GL_n\mathbb{C}$, we obtain a $GL_n\mathbb{C}$-bdle over $BU(n)$. Show that the classifying map for this bdle is a weak htpy equivalence $BU(n) \longrightarrow BGL_n\mathbb{C}$. [You may assume this map exists - e.g. it exists if $BU(n)$ is a CW cplx.]

(Hint: to deal with basepts, use the result in Hatcher §4A.)

d) MS Problem 2-E: Give a direct construction of an isometry b/w two different metrics on the same bdle.

7. Velocity and tangents: Show that for any smooth curve $\gamma: (-\varepsilon, \varepsilon) \to M$ (M a smooth mfld), the velocity vector $\gamma'(0) \in T_{\gamma(0)}M$ is the of the form $D\gamma(u)$, where $u \in T_0(-\varepsilon,\varepsilon) \cong \mathbb{R}$ is a unit vector.

8. Show that if $\begin{array}{ccc} F & \to & E \\ & & \downarrow{\scriptstyle p} \\ & & B \end{array}$ and $\begin{array}{ccc} F' & \to & B \\ & & \downarrow{\scriptstyle p'} \\ & & B' \end{array}$ are fibration sequences, then $\begin{array}{c} E \\ \downarrow{\scriptstyle p' \circ p} \\ B' \end{array}$ is a fibration whose Fiber, F'', sits in a fibration sequence $\begin{array}{ccc} F & \to & F'' \\ & & \downarrow \\ & & F' \end{array}$.