Pontrjagin class (Presentation by Hang Wang) ①

Some notation:

① ω ~~conjugate~~ complex vector bundle, $\begin{cases} \bar{\omega} \text{ conjugate vector bundle} \\ \omega_{\mathbb{R}} \text{ real underlying space} \end{cases}$

② ω real vector bundle, $\omega \otimes_{\mathbb{R}} \mathbb{C}$ complexification

remark: ω $\mathbb{C}$ bundle, $\bar{\omega}$ complex bundle with the same underlying space

$\omega_{\mathbb{R}} = \bar{\omega}_{\mathbb{R}}$ with oposite complex structure $J_1(x,y) = (-y, x)$ $J_2(x,y) = (y, -x)$.

with J_2
$$\begin{array}{ccc} \mathbb{R}\oplus\mathbb{R} & \xrightarrow{id} & \bar{\mathbb{C}} \\ (a,b) & \longmapsto & a-ib \\ \downarrow J_2 & & \downarrow (\cdot i) \\ (b,-a) & \longmapsto & b+ia \end{array}$$

with J_1
$$\begin{array}{ccc} \mathbb{R}\oplus\mathbb{R} & \xrightarrow{id.} & \mathbb{C} \\ (a,b) & \longmapsto & a+ib \\ \downarrow J_1 & & \downarrow i \\ (-b,a) & \longmapsto & -b+ia \end{array}$$

remark $\quad C_k(\bar{\omega}) = (-1)^k C_k(\omega) \qquad C_k(\omega^*) = (-1)^k C_k(\omega)$. have prove?

F complex vector space, fiber in ω, with a Hermition metric $\langle \cdot , \cdot \rangle$ linear, conj. linear

$\bar{F} \cong Hom(F, \mathbb{C}) \cong F^*$

$v \mapsto \langle \cdot , v \rangle$

$vi \mapsto \langle \cdot , iv \rangle = -i \langle \cdot , v \rangle$

Lemma $\quad V \otimes \mathbb{C} = V \oplus iV$ (V real vector space (as fiber of $\mathbb{R}$-vector bundle $E \to X$)) (pf $v \otimes (a+bi) \mapsto (av, ibv)$ $\quad v, a, b \in \mathbb{R}$)

$E \otimes \mathbb{C} \cong \overline{E \otimes \mathbb{C}}$ (pf $x + iy \mapsto x - iy$). $\square$

$$\left.\begin{aligned} c(E\otimes\mathbb{C}) &= 1 + c_1(E\otimes\mathbb{C}) + \cdots + c_n(E\otimes\mathbb{C}) \\ &\parallel \\ c(\overline{E\otimes\mathbb{C}}) &= 1 + c_1(\overline{E\otimes\mathbb{C}}) + \cdots + c_n(\overline{E\otimes\mathbb{C}}) \\ &= 1 - c_1(E\otimes\mathbb{C}) + \cdots + (-1)^n c_n(E\otimes\mathbb{C}) \end{aligned}\right\} \Rightarrow 2 \cdot C_{odd}(E\otimes\mathbb{C}) = 0$$

Def $\quad P_i(E) = (-1)^i C_{2i}(E\otimes\mathbb{C}) \in H^{4i}(X, \mathbb{Z})$. $\qquad P_i(E) = 0$ for $i > \frac{n}{2}$ ($n = \dim E$)

ith Pontrjagin-class

$P(E) = 1 + P_1(E) + \cdots + P_{[\frac{n}{2}]}(E) \in H^*(X, \mathbb{Z})$

Prop ① P is natrual. $f: X \to Y$, $E \to Y$, $f^*: H^*(Y) \to H^*(X)$.

$P(f^*E) = f^*P(E)$. (pf: from natral of c).

② $E \to X$, T (trivial) $\to X$ $\qquad P(T \oplus E) = P(E)$.

(pf: def of p & $c_i(E\otimes\mathbb{C} \oplus T\otimes\mathbb{C}) = c_i(E\otimes\mathbb{C})$).

③. $2\cdot(p(E\oplus F)-p(E)p(F))=0$

$E, F \to X$

Pf $(E\oplus F)\otimes\mathbb{C} = \underline{E\otimes\mathbb{C}}\oplus \underline{F\otimes\mathbb{C}} \Rightarrow c_k((E\oplus F)\otimes\mathbb{C}) = \sum_{i+j=k} c_i(E\otimes\mathbb{C})\, c_j(F\otimes\mathbb{C})$

mod 2, all odd chern disappear

$$\left.\begin{aligned} c_{2k}((E\oplus F)\otimes\mathbb{C}) &\equiv \sum_{\substack{2i+2j=2k\\ i+j=k}} c_{2i}(E\otimes\mathbb{C})\, c_{2j}(F\otimes\mathbb{C}). \\ (-1)^k c_{2k}((E\oplus F)\otimes\mathbb{C}) &\equiv \sum_{i+j=k} (-1)^i c_{2i}(E\otimes\mathbb{C}) (-1)^j c_{2j}(F\otimes\mathbb{C}). \\ p_k(E\oplus F) &\equiv \sum_{i+j=k} p_i(E)\, p_j(F). \end{aligned}\right\} \text{mod } 2.$$

<u>ex</u> S^n $\quad p(TS^n)=1 \quad \Big(TS^n\oplus \underbrace{\underline{NS^n} = S^n\times\mathbb{R}^n}_{\text{trivial.}} \Big).$

<u>Lemma</u> E $\mathbb{C}$ bundle. $E_{\mathbb{R}}\otimes\mathbb{C}\cong E\oplus\bar{E}$

<u>Pf</u> fiber: F (complex vector space)

$V=F_{\mathbb{R}}$. $\quad V\oplus V \leftarrow$ comple structure $J(x,y)=(-y,x)$.

$g: F\longrightarrow V\oplus V\cong F_{\mathbb{R}}\otimes\mathbb{C}$ $\quad g(ix)=J(g(x))$

$x\longmapsto(x,-ix)$.

$$\begin{array}{ccc} x & \longrightarrow & (x,-ix) \\ \downarrow i & & \downarrow J \\ ix & \longmapsto & (ix,x) \end{array}$$

$h: F\to V\oplus V\cong F_{\mathbb{R}}\otimes\mathbb{C}$ $\quad h$

$x\mapsto(x,ix)$

$$\begin{array}{ccc} x & \longrightarrow & (x,ix) \\ \downarrow -i & & \downarrow J \\ -ix & \longrightarrow & (-ix,x) \end{array}$$

$\forall (x,y)\in V\oplus V$ can be written as

$$g\left(\frac{x+iy}{2}\right)+h\left(\frac{x-iy}{2}\right) = \left(\frac{x+iy}{2}+\frac{x-iy}{2}, \frac{-ix+y}{2}+\frac{ix+y}{2}\right)=(x,y)$$

so $F_{\mathbb{R}}\otimes\mathbb{C} = V\otimes\mathbb{C} = V\oplus V \cong g(F)\oplus h(F) \cong F\oplus\bar{F}$ $\qquad\square$.

complex linear $\qquad$ conj. linear.

<u>Prop</u> E $\mathbb{C}$-linear. $\quad p_i=p_i(E_{\mathbb{R}}) \quad c_i=c_i(E)$

then $1-p_1+p_2+\cdots+(-1)^n p_n = (1-c_1+\cdots+(-1)^n c_n)(1+c_1+\cdots+c_n)$

Pf $\text{LHS}=1+c_2(E_{\mathbb{R}}\otimes\mathbb{C})+\cdots+c_{2n}(E_{\mathbb{R}}\otimes\mathbb{C}) = 1+c_1(E_{\mathbb{R}}\otimes\mathbb{C})+\cdots+c_{2n}(E_{\mathbb{R}}\otimes\mathbb{C})$

$c_{odd}(E\oplus\bar{E})=0$

$\sum_{i+j=odd} c_i(E)\, c_j(\bar{E})$ $\leftarrow$ even terms.

$= c(E\oplus\bar{E}) = c(E)\,c(\bar{E}) = \text{RHS}.$

Lemma

$\longrightarrow p_k(E_{\mathbb{R}}) = c_k(E)^2-2c_{k-1}(E)\,c_{k+1}(E)+\cdots+(-1)^k 2c_{2k}(E).$

ex $\tau = T(\mathbb{C}P^n)$

Claim $c(T\mathbb{C}P^n) = (1+a)^{n+1}$ where $a = -c_1(\gamma_1)$ $\qquad$ $\gamma_1 \downarrow \mathbb{C}P^n$ canonical line bundle

so $p_i = p(\tau_{\mathbb{R}})$

$$1 - p_1 + p_2 - \cdots + (-1)^n p_n = c(\tau)c(\bar{\tau}) = (1+a)^{n+1}(1-a)^{n+1} = (1-a^2)^{n+1}$$

$$p_k(\tau_{\mathbb{R}}) = \binom{n+1}{k} a^{2k}$$

<u>Multiplicative sequence</u>:

$\mathbb{Q}[[x]] = \{$formal power series in x with $\mathbb{Q}$ coefficient with constant term $1\}$.

$\forall f(x) \in \mathbb{Q}[[x]]$ $\forall n \in \mathbb{Z}^+$ $f(x_1)\cdots f(x_n)$ symmetric can be expressed by

$$1 + F_1(\sigma_1) + F_2(\sigma_1, \sigma_2) + \cdots + F_n(\sigma_1, \cdots, \sigma_n) + \cdots$$

where $\sigma_k(x_1, \cdots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$ for $1 \le k \le n$

$\sigma_k(x_1, \cdots x_n) = \sigma_k(x_1, \cdots, x_n, 0, \cdots, 0)$ for $k > n$

and F is homogeneous of degree k (i.e. $F_k(t\sigma_1, \cdots, t^k\sigma_k) = t^k F(\sigma_1, \cdots, \sigma_k)$ $\forall t \in \mathbb{Q}$)

<u>Def</u> The sequence of polynomials $\{F_k(\sigma_1, \cdots, \sigma_k)\}_{k=1}^{\infty}$ is called multiplicative sequence determined by $f(x)$.

Let B be a commutative graded algebra over $\mathbb{Q}$, with decomposition

$B = B^0 \oplus B^1 \oplus \cdots \oplus B^n \oplus \cdots$ $\quad$ (s.t. $B^i \cdot B^j \subseteq B^{i+j}$ $\forall i, j$).

Denote $B^\wedge$ the set of all formal sum $1 + b_1 + \cdots + b_n + \cdots$ ($b_k \in B^k$)

the finite sum is closed under multiplication and can be extended to $B^\wedge$

by define $(1 + b_1 + b_2 + \cdots)(1 + c_1 + c_2 + \cdots) = 1 + (b_1 + c_1) + (b_1c_1 + b_2 + c_2) + \cdots$

For a multiplicative sequence $\{F_k(\sigma_1, \cdots, \sigma_k)\}_{k=1}^{\infty}$ Then to algebra B we associative a map $F: B^\wedge \to B^\wedge : b = 1 + b_1 + b_2 + \cdots \mapsto 1 + F_1(b_1) + F_2(b_1, b_2) + \cdots$

<u>Lemma</u> $F: B^\wedge \to B^\wedge$ satisfy $F(bc) = F(b)F(c)$ $\forall b, c \in B^\wedge$

pf (Lawson & Michelsohn, Spin Geometry P229).

Construction of multiplicative sequence of Pontrjagin class.

Let $\{F_k\}$ be multiplicative sequ. associ. to formal power series $f(x)$

To each real vector bundle E over X we associate total F-class

by $F(E) \equiv F(p(E)) \in H^{4*}(X, \mathbb{Q})$.

$E_1, E_2 \to X$ $\quad p(E_1 \oplus E_2) = p(E_1) \oplus p(E_2) \pmod 2 \overset{\mathbb{Q}\text{ coefficient kill the torsion}}{\Rightarrow} F(E_1 \oplus E_2) = F(E_1) F(E_2)$

Assume E is oriented and of dim $2n$. By spliting principal (Spin Geometry 226) ~~we have~~

$\exists \pi : P \to X$ s.t. $\pi^* E = E_1 \oplus \cdots \oplus E_n$ $\dim_{\mathbb{R}} E_n = 2$ (real) E_n oriented. and

$\pi^* E \otimes \mathbb{C} = l_1 \oplus \bar{l}_1 \oplus l_2 \oplus \bar{l}_2 \oplus \cdots \oplus l_n \oplus \bar{l}_n$ $\quad l_i$ $\mathbb{C}$-line bundle with $E_i \otimes \mathbb{C} \cong l_i \oplus \bar{l}_i$.

$\pi^* F(E) = F(\pi^* E) \underset{\text{multiplicative}}{=} F(E_1) \cdots F(E_n) = F(p(E_1)) \cdots F(p(E_n)) = F(1 + p_1(E_1)) \cdots F(1 + p_1(E_n))$

$\underset{\text{MS §19}}{=} f(p_1(E_1)) \cdots f(p_1(E_n)) = f(x_1) \cdots f(x_n)$ where $x_i = p_1(E_i)$

ex $\quad f(x) = 1 + x \quad$ Let $E = E_1 \oplus \cdots \oplus E_n$ ($\mathbb{R}$-bundle dim $2n$) $\dim_{\mathbb{R}} E_n = 2$ (real.) $\quad x_i = p_1(E_i)$

then $F(E) = f(x_1) \cdots f(x_n) = (1 + x_1) \cdots (1 + x_n) = 1 + \sigma_1 + \cdots + \sigma_n = 1 + p_1(E) + \cdots + p_n(E)$

$= p(E)$.

Rmk Multiplicative sequence is a way to build new char. class from existing ones.

Rmk we can construct class for complex bundle from chern class.

ex

$l(x) = \dfrac{\sqrt{x}}{\tanh\sqrt{x}} \in \mathbb{Q}[[x]]$. $\quad E$ oriented. $2n$ dim, $\mathbb{R}$-bundle. over X.

Define a L-class:

$\pi^* E = E_1 \oplus \cdots \oplus E_n$ by splitting principle

$\dim_{\mathbb{R}} E_i = 2$ oriented.

$\pi^* L(E) = F(\pi^* E) = l(x_1) \cdots l(x_n) = 1 + L_1(\sigma_1) + L_2(\sigma_1, \sigma_2) + \cdots$

(Let $x_i = p_1(E_i)$ then $\sigma_i = p_i(\pi^* E) = \pi^* p_i(E)$)

$= 1 + L_1(p_1) + L_2(p_1, p_2) + \cdots = \pi^*(1 + L_1(p_1) + \cdots + L_n(p_1, \cdots, p_n) + \cdots)$

$L(E) = 1 + L_1(p_1) + L_2(p_1, p_2) + \cdots + L_n(p_1, \cdots, p_n) + \cdots$ since π^* injective

Def $\quad$ L-genus $\quad L[M^m] = \begin{cases} 0 & m \neq 4k \\ \langle L_k(p_1(TM), \cdots, p_k(TM)), [M^{4k}] \rangle & m = 4k. \end{cases}$

Def $\quad$ Signature of compact oriented mfd M^n is defined to be

0 when $n \neq 4k$

when $n = 4k$, Let $a_1, \ldots, a_m$ be basis for $H^{2k}(M^{4k}, \mathbb{Q})$

$a_{ij} = \langle a_i \cup a_j, [M^{4k}] \rangle$ $A = (a_{ij})_{m\times m}$ is symmetric

signature of M^{4k} is defined as signature of A, denoted by $\sigma(M^{4k})$.

(Recall, signature of a symmetric matrix A: $\exists\, C$ s.t. $C^T A C = D \leftarrow$ diagonal matrix. $\operatorname{sign}(A)$ = number of the positive numbers in the diagonal − number of the negative numbers in the diagonal.)

Hirzebruch Signature Thm (MS §19): For M^{4k} real, oriented mfd. we have $\sigma(M^{4k})$ = L-genus of M.

$\sigma(M^{4k})$: signature of the mfd M^{4k}

L-genus of M: $= L[M^{4k}] = \langle L_k(p_1, \cdots, p_k), [M^{4k}] \rangle$.