Proof of the Projective Bdle Theorem:

First we recall the statement.

Theorem: Let $\begin{matrix} \mathbb{C}^n \to E \\ \downarrow \\ B \end{matrix}$ be a cplx vector bdle, with B paracpt, and let $\begin{matrix} P(E) \\ \downarrow q \\ B \end{matrix}$ be the corresponding projective space bdle.

Then the map $q^*: H^*(B;\mathbb{Z}) \to H^*(PE;\mathbb{Z})$ is injective, and the induced map

$$H^*(B) \otimes_{\mathbb{Z}} \underbrace{\mathbb{Z}[a, a^2, \ldots, a^{n-1}]}_{\substack{\text{graded free abelian} \\ \text{group on } n \text{ generators,} \\ |a^i| = 2i}} \longrightarrow H^*(PE;\mathbb{Z})$$

sending $a^i \longrightarrow c_1(L_E)^i$ is an isomorphism of H^*B-modules. The same result holds for real bdles, with $\mathbb{Z}$ replaced by $\mathbb{Z}/2$.

Proof: We begin by proving the theorem for CW cplxs B, starting with finite dim'l complexes. If B is zero-dim'l, then $P(E) = \coprod_{b \in B} \mathbb{C}P^{n-1}$, so the result follows from our computation of $H^*(\mathbb{C}P^{n-1};\mathbb{Z})$, and the fact that the restriction of L_E to each copy of $\mathbb{C}P^{n-1}$ is the tautological bdle $\begin{matrix} \gamma_1^{n-1} \\ \downarrow \\ \mathbb{C}P^{n-1} \end{matrix}$

Now assume the result for $(k-1)$-dim'l complexes, and say B is k-dim'l. We will consider the following diagram, in which $B' \subseteq B$ denotes the subspace formed by removing one point from the interior of each k-cell, and we replace the polynomial ring by $H^*\mathbb{C}P^{n-1}$:

$$(\star)\quad\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^{*-1}(q^{-1}B') \arrow[r, "\partial"] & H^*(PE, q^{-1}B') \arrow[r, "j^*"] & H^*(PE) \arrow[r, "i^*"] & H^*(q^{-1}B') \arrow[r] & \cdots \\
& H^{*-1}(B') \otimes_{\mathbb{Z}} H^{*-1}\mathbb{C}P^{n-1} \arrow[r, "\partial \otimes id"] \arrow[u, "\Phi'"] & H^*(B,B') \otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[r, "j^* \otimes id"] \arrow[u, "\Phi^{rel}"] & H^*(B) \otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[r, "i^* \otimes id"] \arrow[u, "\Phi"] & H^*(B') \otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[r] \arrow[u, "\Phi'"] & \cdots
\end{tikzcd}$$

The top row of $(\star)$ is just the LES of the pair $(PE, q^{-1}B')$. The bottom row requires some explanation. We use $H^* q^{-1}B' \otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1}$ as short-hand for the $*$-th graded piece of the graded tensor product; this means that the lower LES is actually the <u>sum</u> of various shifted copies of the LES for the pair (B, B'); here we use that $H^*(\mathbb{C}P^{n-1})$ is always either $\mathbb{Z}$ or 0.

The vertical maps Φ, Φ' are induced by the map of pairs

$$(PE, q^{-1}B') \xrightarrow{q} (B, B')$$

and the assignments $\alpha^i \mapsto c_1(L_E)^i$ or $\alpha^i \mapsto j^*(c_1 L_E)$ ($\alpha \in H^2\mathbb{C}P^n$ is the canonical generator). The map Φ^{rel} is defined similarly, via the relative cup product:

$$H^*(B,B') \otimes H^*\mathbb{C}P^{n-1} \xrightarrow{q^* \otimes (\alpha \mapsto c_1 L_E)} H^*(PE, q^{-1}B') \otimes H^* PE \xrightarrow{\cup} H^*(PE, q^{-1}B').$$

Commutativity of the middle and right squares is immediate, and we must check commutativity of the left square. We must show that for any $x_i \in H^{*-1}B'$, We must show that for any $x_k \in H^{*-1}B'$,

$$\Phi^{rel}\partial\left(\sum_k x_k \otimes \alpha^k\right) = \partial\Phi'\left(\sum_k x_k \otimes \alpha^k\right).$$

We have

$$\Phi^{rel}\partial\left(\sum x_k \otimes \alpha^k\right) = \Phi^{rel}\left(\sum (\partial x_k) \otimes \alpha^k\right) = \sum q^*(\partial x_k) \cup (c_1 L_E)^k$$
$$= \sum \partial q^* x_k \cup (c_1 L_E)^k$$

and

$$\partial\Phi'\left(\sum x_k \otimes \alpha^k\right) = \partial\left(\sum (q^* x_k) \cup i^*(c_1 L_E)^k\right) = \sum \partial(q^* x_k \cup i^*(c_1 L_E)^k).$$

So we need to show that $\partial(q^* x_k \cup i^*(c_1 L_E)^k) = \partial(q^* x_k) \cup (c_1 L_E)^k$.

<u>Lemma</u>: For any pair (X, A), the bdry map in the LES

$$H^{*-1}X \xrightarrow{i^*} H^{*-1}A \xrightarrow{\partial} H^*(X,A) \to H^*X$$

satisfies $\partial(\alpha \cup i^* x) = \partial\alpha \cup x$.

<u>Proof</u>: The map ∂ is defined as follows: if $\tilde{a} \in C^{*-1}X$ is any cochain restricting to a cocycle $a \in C^*A$, then $\partial([a]) := [\delta(\tilde{a})]$, which is a cochain vanishing on C_*A, i.e. $\delta(\tilde{a}) \in C^*(X,A) \subseteq C^*X$.

Now, given $x = [x] \in H^{*-1}X$ and $\alpha = [a] \in H^{*-1}A$, choose $\tilde{a} \in C^{*-1}X$ extending a. Then $\tilde{a} \cup x \in C^{2*-2}X$ restricts to $a \cup i^* x$, so

$$\partial(\alpha \cup i^* x) = \partial[a \cup i^* x] = \delta(\tilde{\alpha} \cup x)$$
$$= (\delta\tilde{\alpha}) \cup x \pm \tilde{\alpha} \cup \delta x = \delta\tilde{\alpha} \cup x. \quad \square$$

$\boxed{\delta \text{ is a derivation}}$ $\boxed{x \text{ is a cocycle so } \delta x = 0}$

By the 5-lemma, to complete the argument for fin.dim'l CW base spaces, it suffices to prove:

Claim 1: Φ' is an isomorphism.

Claim 2: Φ^{rel} is an isomorphism

Proof of Claim 1: We have a commutative diagram of fibrations

$$\begin{tikzcd} \mathbb{C}P^{n-1} \arrow[r, equal] \arrow[d] & \mathbb{C}P^{n-1} \arrow[d] \\ q^{-1}(B^{(k-1)}) \arrow[r, hook] \arrow[d, "q"] & q^{-1}(B') \arrow[d, "q"] \\ B^{(k-1)} \arrow[r, hook] & B' \end{tikzcd}$$

. Since $D^k - \{pt\}$ deformation retracts to S^{n-1}, B' deformation retracts to $B^{(k-1)}$.

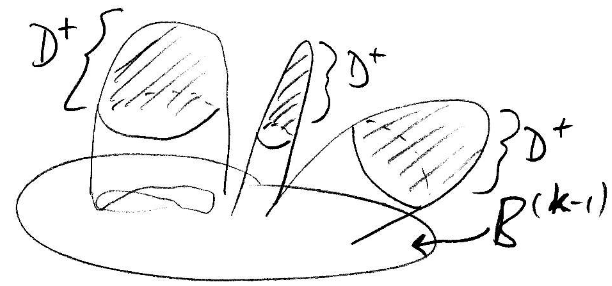
The 5-lemma now implies that $q^{-1}(B^{(k-1)}) \hookrightarrow q^{-1}(B')$ induces an isomorphism on π_*, and hence on H^* (by Hatcher Prop. 4.21). The diagram

$$\begin{tikzcd} H^* q^{-1}(B^{(k-1)}) \arrow[r, "\cong"] & H^*(q^{-1}B') \\ H^*B^{(k-1)} \otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[r, "\cong"] \arrow[u, "\Phi^{(k-1)}"] & H^*B' \otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1} \arrow[u, "\Phi'"] \end{tikzcd}$$

completes the proof of Claim 1, since by induction Φ^{k-1} is an isomorphism. $\square$

Proof of Claim 2:

Let $D^+ \subseteq B$ denote the disjoint union of the closed, radius $\frac{1}{2}$ disks inside each n-cell in B:



D^+ = shaded region

Then D^+ deformation retracts to the disjoint union of points $B - B'$, and $D^+ \cap B'$ deformation retracts to the disjoint union of the boundary $(k-1)$-spheres of the various disks in D^+. Note that these are both CW cplxs of dim'n $< k$, so the theorem applies to them by induction.

We have excision isomorphisms $H^*(B, B') \cong H^*(D^+, D^+ \cap B')$ and $H^*(PE, PE') \cong H^*(PE|_{D^+}, PE|_{D^+ \cap B'})$, coming from excising the complements of D^+ and of $PE|_{D^+}$ respectively. This gives a comm. diagram

$$\begin{tikzcd}
H^*(PE, PE') \arrow[r, "\cong"] & H^*(PE|_{D^+}, PE|_{D^+ \cap B'}) \\
H^*(B,B') \otimes H^*\mathbb{C}P^{n-1} \arrow[u] \arrow[r, "\cong"] & H^*(D^+, D^+ \cap B') \otimes H^*\mathbb{C}P^{n-1} \arrow[u]
\end{tikzcd}$$

and hence it suffices to prove that the right hand map is an isom.

But this follows by applying the induction hypothesis to the LES's of the pairs $(PE|_{D^+}, PE|_{D^+\cap B'})$, $(D^+, D^+\cap B')$: we have a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{*-1}(PE|_{D^+\cap B}) \arrow[r, "\partial"] & H^*(PE|_{D^+}, PE|_{D^+\cap B'}) \arrow[r] & H^*PE|_{D^+} \arrow[r] & H^*PE|_{D^+\cap B'} \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{*-1}(D^+\cap B)\otimes H^*\mathbb{C}P^{n-1} \arrow[r, "\partial"] \arrow[u] & H^*(PE|_{D^+}, PE|_{D^+\cap B'})\otimes H^*\mathbb{C}P^{n-1} \arrow[r] \arrow[u, "\Phi^{rel}"] & H^*PE|_{D^+}\otimes H^*\mathbb{C}P^{n-1} \arrow[r] \arrow[u] & H^*(D^+\cap B)\otimes H^*\mathbb{C}P^{n-1} \arrow[u] &
\end{tikzcd}$$

and the argument in the proof of Claim 1 shows that the vertical arrows other than Φ^{rel} are isom's. The 5-Lemma completes the proof. $\square$

We have now shown that the Proj. Bdle Thm holds for any finite dim'l CW cplx.

If B is a possibly infinite-dim'l CW complex, then we need to show that the map

$$\Phi : H^*B\otimes H^*\mathbb{C}P^{n-1} \overset{def}{=} \bigoplus_{\substack{*-k \text{ even} \\ *-k>0}} H^kB\otimes \underbrace{H^{*-k}\mathbb{C}P^{n-1}}_{\mathbb{Z}} \longrightarrow H^*PE$$

is an isomorphism. If we choose $N > *+1$, then by the cellular approximation theorem we have $\pi_i B^{(N)} \xrightarrow{\cong} \pi_i B$ for $i \le N-1$, and in particular for $i \le *+1$. Hence by [Hatcher 4.21], $B^{(N)} \hookrightarrow B$ (and consequently $PE|_{B^{(N)}} \hookrightarrow PE$) induce isom's on H^k for $k \leq *$. The result now follows from the diagram

$$\begin{tikzcd} H^* PE|_{B^{(N)}} & H^* PE \arrow[l, "\cong"'] \\ H^* B^{(N)} \otimes H^* \mathbb{C}P^{n-1} \arrow[u, "\cong"] & H^* B \otimes H^* \mathbb{C}P^{n-1} \arrow[l, "\cong"'] \arrow[u] \end{tikzcd}$$

Finally, consider an arbitrary paracompact base space B. Then as shown in Hatcher Prop'n 4.13 there exists a "CW approximation" $K \xrightarrow{f} B$, i.e. K is a CW cplx and $f_* : \pi_* K \to \pi_* B$ is an isom. for all $*$ (hence $H^* B \to H^* K$ is also an isom.).

The diagram

$$\begin{tikzcd} H^* PE \arrow[r, "\cong"] & H^* (P(f^* E)) \\ H^* B \otimes H^* \mathbb{C}P^n \arrow[u] \arrow[r, "\cong"] & H^* K \otimes \mathbb{C}P^n \arrow[u, "\cong"'] \end{tikzcd}$$

now completes the proof; the top map comes from the isomorphism $P(f^* E) \cong f^* P(E)$ and the diagram
$$\begin{tikzcd} f^* PE \arrow[r] \arrow[d] & PE \arrow[d] \\ K \arrow[r, "f"] & B \end{tikzcd}$$
(the top map is an isom on π_* and hence H^* by the 5-lemma). $\square$

<u>Rmk</u>: Where did we use paracompactness of the base space B??. For this argument to make sense, we need to know that the class $c_1(L_E) \in H^* PE$ <u>is defined</u>. This means the line bdle $\begin{smallmatrix} L_E \\ \downarrow \\ PE \end{smallmatrix}$ must admit a classifying map

$$\begin{tikzcd} L_E \arrow[r, "F"] \arrow[d] & \gamma_1 \arrow[d] \\ PE \arrow[r, "f"] & \mathbb{C}P^\infty \end{tikzcd}$$

, which we can only guarantee if B is paracompact [MS §5, or Hatcher's VB notes, p. 79]

On the other hand, in the real case we saw that $w_1\left(\begin{smallmatrix} L \\ \downarrow \\ X \end{smallmatrix}\right) \in H^1(X; \mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1 X, \mathbb{Z}/2)$ could be defined over <u>any</u> base space. So we don't need paracptness when the bundle is a <u>real</u> vector bdle.