Lecture 19 Bott Periodicity III: Linear Clutching Fcns

We have now shown that every bdle $[E,f]$ over $X \times S^2$ has the form $[E, \ell]$ for some Laurent poly. clutching fcn $\ell(x,z) = \sum_{-N}^{N} a_n(x) z^n$.

We now have $\left[E, \sum_{-N}^{N} a_n(x) z^n\right] = \left[E, z^{-N}\left(\sum_{n=-N}^{N} a_n(x) z^{n+N}\right)\right]$

$$= \Big[E, \sum_{n=0}^{2N} \underbrace{a_n'(x)}_{a_n' = a_{n-N}} z^n\Big] \otimes [\varepsilon', z^{-N}] = [E, q] \otimes \pi_2^*(H^{-N})$$

We write $[\varepsilon', z^{-N}] = H^{-N}$ b/c $[\varepsilon', z^{-N}] \otimes \underbrace{[\varepsilon', z^{N}]}_{H^N} = [\varepsilon', 1] =$

where $q = \sum_{n=0}^{2N} a_n'(x) z^n$ is a <u>polynomial</u> clutching fcn.

<u>Prop. 2.6</u>: If $q = \sum_{n=0}^{N} q_n(x) z^n$ is a poly. clutching fcn, then $[E,q] \oplus [nE, \mathbb{1}] \cong [(n+1)E, L^N q]$ for some <u>linear</u> clutching fcn $L^N q = a_0' + a_1' z$.

<u>Pf</u>: $[E,q] \oplus [NE, \mathbb{1}] \cong \left[(N+1)E, \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & q \end{bmatrix}\right]$, and the matrices $\begin{bmatrix} 1 & -z & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & -z \\ a_N & \cdots & & a_0 \end{bmatrix}$, $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 & \\ & & & q \end{bmatrix}$ are equiv under row and column op's:

Linear in z as an endomorphism $(N+1)E \times S^1 \to (N+1)E \times S^1$

$$\begin{bmatrix} 1 & -z & & 0 \\ & 1 & \ddots & \\ 0 & & 1 & -z \\ a_N & \cdots & & a_0 \end{bmatrix} \overset{c_2 \mapsto c_2 + z c_1}{\sim} \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & -z & \\ & & \ddots & -z \\ a_N & z a_N + a_{N-1} & \cdots & a_0 \end{bmatrix} \overset{c_3 \mapsto c_3 + z c_2}{\sim} \cdots \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ a_N & z a_N + a_{N-1} & \cdots & q \end{bmatrix} \sim \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & q \end{bmatrix}$$

Each op'n is a htpy: for example, the first can be viewed as $c_2 \mapsto c_2 + t z c_1$ w/ $t = 1$. So it remains to check that the first matrix is inv'ble. But in each fiber, choosing a basis we can view these row/col. op's as "block" op'ns, i.e. k op'ns if fiber dim'n is k, and each multiplies det by a non-zero scalar. So since q is inv'ble, so is 1st matrix. ☐

Next time, we'll prove

<u>Prop. 2.7</u> (Hatcher)

For any $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ and any clutching function $az+b$, with $a, b : E \to E$ bundle endomorphisms, there exists a direct sum decomposition $E = E_+ \oplus E_-$ s.t.

$$[E, az+b] \cong [E_+, \overset{\text{identity c.f.}}{\mathbb{1}}] \oplus [E_-, z]$$
$$= \pi_1^*(E_+) \oplus [E_-, \mathbb{1}] \otimes [\varepsilon', z]$$
$$= \pi_1^* E_+ \oplus \left[(\pi_1^* E_-) \otimes \pi_2^* H\right].$$

Let's examine the consequences of this result in K-theory:

We've shown that for any bdle $\begin{matrix} E' \\ \downarrow \\ X \times S^2 \end{matrix}$, we have

(1) $$E' \cong [E, \overset{\text{LPCF}}{\ell}] \cong [E, \overset{\text{Polynomial c.f.}}{q}] \otimes [\varepsilon', z^{-N}] = [E, q] \otimes \pi_2^*(H^{-N})$$

(Here N = degree of the Laurent Poly. ℓ, and q has degree $2N$ (b/c $q = z^N \ell$).

(2) Since $[E, q] \oplus \underbrace{[2NE, \mathbb{1}]}_{2N \cdot \pi_1^* E} \cong [(2N+1)E, \overset{\text{endomorphisms } E \to E}{az+b}]$

Let $(2N+1)E = \tilde{E}_+ \oplus \tilde{E}_-$ be the direct sum decomposition guaranteed by Prop 2.7, so that

(3) $$[(2N+1)E, az+b] \cong \pi_1^*(\tilde{E}_+) \oplus \left(\pi_1^*(\tilde{E}_-) \otimes \pi_2^* H\right).$$

Then in $K^0(X \times S^2)$, we have (combining (2) and (3))

(4) $$[E, q] = [\pi_1^*(\tilde{E}_+)] + [\pi_1^*(\tilde{E}_-)] \cdot [\pi_2^* H] - 2N[\pi_1^* E].$$

Finally, (1) and (4) yield

$$[E'] = \left([\pi_1^*(\tilde{E}_+)] + [\pi_1^*(\tilde{E}_-)] \cdot [\pi_2^* H] - 2N[\pi_1^* E]\right) \cdot \pi_2^* H^{-N}.$$

The expression on the right lies in the image of

$$K^0(X) \otimes K^0(S^2) \xrightarrow{\mu} K^0(X \times S^2),$$
$$\alpha \otimes \xi \longmapsto \pi_1^* \alpha \otimes \pi_2^* \xi$$

so we conclude that μ is surjective. In fact, the map

$$K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K^0(X) \otimes K^0(S^2) \xrightarrow{\mu} K^0(X \times S^2)$$
$$H \longmapsto [H] = [\gamma_1]$$

is surjective as well, because H is invertible in the ring $\mathbb{Z}[H]/(H-1)^2$:
$$(2-H)H = 2H - H^2$$
$$= 2H - (2H-1)$$
$$= 1$$

so $2-H = H^{-1}$.

Letting $X = pt$, we have:

<u>Theorem</u>: $K^0(S^2) \cong \mathbb{Z}[H]/_{(H-1)^2}$, via the ring map sending $H \longmapsto [H] = [\gamma_1^1]$.

<u>Pf</u>: We already showed that this map exists and the previous discussion shows that it is surjective. To show injectivity, note that any element in $\mathbb{Z}[H]/_{(H-1)^2}$ has the form $nH + m$ for some $n, m \in \mathbb{Z}$. If $nH + m \longmapsto 0$ in $K^0(S^2)$, then $n[H] + m[\varepsilon^1] = 0$, which implies that $m = -n$ (since this class must restrict to zero in $K^0(\{x\})$ for any $x \in S^2$, meaning nH and $-m[\varepsilon^1]$ must have the same dimension).

Now, if $n[H] - [\varepsilon^m] = 0$ in $K^0(S^2)$, then by definition of K^0 we have $nH \oplus F \cong \varepsilon^m \oplus F$ for some bundle $[F]$. But in fact, we can always take $F = \varepsilon^k$ for some k, b/c $\exists F^\perp$ with $F \oplus F^\perp \cong \varepsilon^k$ (as we showed earlier). So $nH \oplus \varepsilon^k \cong \varepsilon^{m+k}$, and the Whitney Sum Formula implies that

$$nc_1H = c_1(nH) = c_1(nH \oplus \varepsilon^k) = c_1(\varepsilon^{m+k}) = 0.$$

But $c_1H \in H^2(S^2) \cong \mathbb{Z}$ is non-zero, so $n(c_1H) \neq 0$ as well. This contradiction implies that our map is injective. $\square$