Our next goal will be to construct universal principal G-bdles $\begin{matrix} EG \\ \downarrow \\ BG \end{matrix}$. Some of this material is in MS §5.

General Constructions:

- For any group G, Milnor constructs a bdle $\begin{matrix} EG \\ \downarrow \\ BG \end{matrix}$ where $EG = \overset{\infty}{\underset{i=1}{*}} G^{(i)}$ ("infinite join").
  Topologized properly, $EG \simeq *$ and $\begin{matrix} EG \\ \downarrow \\ BG = EG/G \end{matrix}$ is a principal G-bdle. This construction is functorial in G. Moreover, it classifies bdles over all paracpt Hausd. spaces, not just CW cplxs.
- Unclear when this BG is a CW cplx.
- Related constructions: Segal, Milgram.

We'll consider specific constructions for the gps of interest to us: $GL_n\mathbb{R}$, $O(n)$, $GL_n\mathbb{C}$, $U(n)$. The cplx case is nearly identical to the real case, so we'll work over $\mathbb{R}$ (and note the differences).

Def'n: The Stiefel manifold $V_n(\mathbb{R}^{n+k}) \subseteq \underbrace{\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}}_{n}$ is the subspace consisting of all linearly indep. n-tuples $(\vec{x}_1, \dots, \vec{x}_n)$. (Note that $V_n(\mathbb{R}^{n+k})$ is open in $(\mathbb{R}^{n+k})^n$.) Such n-tuples are called n-frames.

The Grassmannian $G_n(\mathbb{R}^{n+k})$ is the quotient space $V_n(\mathbb{R}^{n+k}) \Big/ \begin{matrix} (\vec{x}_1, \dots, \vec{x}_n) \sim (\vec{y}_1, \dots, \vec{y}_n) \\ \text{if } \operatorname{Span}(\{\vec{x}_i\}) = \operatorname{Span}(\{\vec{y}_i\}) \end{matrix}$. Hence $Gr_n(\mathbb{R}^{n+k})$ consists of all n-dim'l subspaces of $\mathbb{R}^{n+k}$. Note that $GL_n\mathbb{R}$ acts on $V_n(\mathbb{R}^{n+k})$ by $\underbrace{(\vec{x}_1, \dots, \vec{x}_n)}_{X} \cdot \underset{(\vec{a}_1 \cdots \vec{a}_n)}{A} = (X\vec{a}_1 \cdots X\vec{a}_n)$ and $V_n(\mathbb{R}^{n+k})/GL_n\mathbb{R} \cong Gr_n(\mathbb{R}^{n+k})$.

There is a second version of the Stiefel mfld, involving orthonormal (o.n.) frames:

<u>Def'n:</u> $V_n^o(\mathbb{R}^{n+k}) = \{(\vec{x}_1, \dots, \vec{x}_n) \in (\mathbb{R}^{n+k})^n \mid \langle x_i, x_j \rangle = \delta_{ij}\} \subseteq (\mathbb{R}^{n+k})^n$

Note that $O(n)$ acts on $V_n^o(\mathbb{R}^{n+k})$ and (again $V_n^o(\mathbb{R}^{n+k})/O(n) \cong Gr_n\mathbb{R}^{n+k}$.
(The inclusion $V_n^o\mathbb{R}^{n+k} \hookrightarrow V_n\mathbb{R}^{n+k}$ induces a cont. bij. $V_n^o\mathbb{R}^{n+k}/O(n) \xrightarrow{\alpha} Gr_n\mathbb{R}^{n+k}$; it's a homeomorphism b/c the Gram-Schmidt process gives a cont. map $V_n\mathbb{R}^{n+k} \xrightarrow{GS} V_n^o\mathbb{R}^{n+k}$ and α^{-1} is given by

$$\begin{tikzcd} V_n\mathbb{R}^{n+k} \arrow[r, "GS"] \arrow[d] & V_n^o\mathbb{R}^{n+k} \arrow[d] \\ Gr_n\mathbb{R}^{n+k} \arrow[r, dashed, "\alpha^{-1}"] & V_n^o(\mathbb{R}^{n+k})/O(n). \end{tikzcd}$$

<u>Def'n:</u> The infinite Stiefel mflds $V_n(\mathbb{R}^\infty)$, $V_n^o(\mathbb{R}^\infty)$ are defined by $V_n(\mathbb{R}^\infty) = \operatorname{colim}_{k\to\infty} V_n(\mathbb{R}^{n+k})$, $V_n^o(\mathbb{R}^\infty) = \operatorname{colim}_{k\to\infty} V_n^o(\mathbb{R}^{n+k})$, and $Gr_n(\mathbb{R}^\infty) := V_n(\mathbb{R}^\infty)/GL_n\mathbb{R} \cong V_n^o(\mathbb{R}^\infty)/O(n) \cong \operatorname{colim}_{n\to\infty} Gr_n(\mathbb{R}^{n+k})$.

Here $\operatorname{colim}(X_1 \hookrightarrow X_2 \hookrightarrow \cdots)$ is topologized by declaring $U \subset \operatorname{colim}_i X_i$ to be open $\iff$ $U \cap X_i$ is open for all i.

<u>Theorem 1:</u> The bundles $\begin{array}{c} V_n(\mathbb{R}^\infty) \\ \downarrow \\ Gr_n(\mathbb{R}^\infty) \end{array}$ and $\begin{array}{c} V_n^o(\mathbb{R}^\infty) \\ \downarrow \\ Gr_n(\mathbb{R}^\infty) \end{array}$ are universal principal $GL_n\mathbb{R}$ / $O(n)$ bdles. Moreover, $V_n(\mathbb{R}^{n+k})$ and $V_n^o(\mathbb{R}^{n+k})$ are $(k-2)$-universal [i.e. $\pi_* V_n\mathbb{R}^{n+k} = \pi_* V_n^o\mathbb{R}^{n+k} = 0$ for $* \leq k-1$.]

(MS66) <u>Theorem 2:</u> $Gr_n(\mathbb{R}^{n+k})$ and $Gr_n(\mathbb{R}^\infty)$ are CW cplxs.

[Note: In the cplx case, Thm 1 would state that these bdles are $(2k-1)$-universal, i.e. $\pi_* V_n^o\mathbb{C}^{n+k} \cong \pi_* V_n\mathbb{C}^{n+k} = 0$ for $* \leq 2k$.]

To prove Theorem 1, we'll use:

Lemma (HW 1): If G acts freely on P, then $\begin{array}{c} P \\ \downarrow{\scriptstyle \pi} \\ P/G \end{array}$ is a principal G-bdle $\Longleftrightarrow$ 1) π admits local sections

2) $$P \times_{P/G} P \longrightarrow G \quad \text{is continuous.}$$
$$(p, p') \longmapsto \left(\begin{array}{c}\text{the unique } g \in G \\ \text{s.t. } p \cdot g = p'\end{array}\right)$$

To find local sections, we'll use the following open cover of $Gr_n(\mathbb{R}^{n+k})$:

Def'n: Given any frame $F \in V_n(\mathbb{R}^{n+k})$, we set
$\mathcal{O}_F = \{ W \in Gr_n(\mathbb{R}^{n+k}) \mid W \cap \mathrm{Span}(F)^{\perp} = \{0\} \}$. (We allow $n+k = \infty$)

Claim: Each $\mathcal{O}_F$ is open, and $\{ \mathcal{O}_F \mid F \in V_n(\mathbb{R}^{n+k}) \}$ covers $Gr_n(\mathbb{R}^{n+k})$ for any $k \leq \infty$

Pf: For any $W \in Gr_n \mathbb{R}^{n+k}$, and any o.n. basis $(w_1, \dots, w_n)$ for W, we have $W \cap \mathrm{Span}(w_1, \dots, w_n)^{\perp} = W \cap W^{\perp} = \{0\}$ so $W \in \mathcal{O}_{(w_1, \dots, w_n)}$.

To see that $\mathcal{O}_F$ is always open, note that its inverse image in $V_n(\mathbb{R}^{n+k})$ is $\{ H \in V_n \mathbb{R}^{n+k} \mid \mathrm{Span}\, H \cap (\mathrm{Span}\, F)^{\perp} = \{0\} \}$.

Now, if $H = \{\vec{h}_1, \dots, \vec{h}_n\}$ and $(\vec{f}_1, \dots, \vec{f}_k)$ is a basis for $(\mathrm{Span}\, F)^{\perp}$, then $\mathrm{Span}\, H \cap (\mathrm{Span}\, F)^{\perp} = \{0\} \Longleftrightarrow \det(\vec{h}_1 \cdots \vec{h}_n \ \vec{f}_1 \cdots \vec{f}_k) \neq 0$.

(When $k = \infty$, this shows that $\mathcal{O}_F \cap Gr_n(\mathbb{R}^{n+k})$ is open in $Gr_n(\mathbb{R}^{n+k})$ for each $k < \infty$, which implies $\mathcal{O}_F$ is open in $Gr_n(\mathbb{R}^{\infty})$.)

Claim: The projections $\begin{array}{ccc} V_n(\mathbb{R}^{n+k}) & & V_n^0(\mathbb{R}^{n+k}) \\ & \pi\searrow \quad \swarrow \pi^0 & \\ & Gr_n(\mathbb{R}^{n+k}) & \end{array}$ both admit sections over each open set $\mathcal{O}_F$ (w/ $F \in V_n^0(\mathbb{R}^{n+k})$ in the case of π.

[Rmk: A thm of Andrew Gleason says that actions of cpt Lie gps always admit slices.] (the space must be regular)

Pf: Given $F = (\vec{x}_1, \dots, \vec{x}_n)$, set $W = \mathrm{Span}(\{x_i\})$.

Then if $V \in \mathcal{O}_F$, elementary linear algebra shows that the orthogonal projection $\mathrm{proj}_{V,W} : W \longrightarrow V$ is a linear isomorphism. We define $s_F(V) = \left(\mathrm{proj}_{V,W}(\vec{x}_i)\right)_{i=1}^n \in V_n\mathbb{R}^{n+k}$, and we define $\bar{s}_F(V) = GS\left(\left(\mathrm{proj}_{V,W}\vec{x}_i\right)_{i=1}^n\right) \in V_n^0(\mathbb{R}^{n+k})$, where GS denotes the Gram-Schmidt orthogonalization process.

Then $\pi s_F = \pi^0 s_F^0 = \mathrm{id}_{Gr_n\mathbb{R}^{n+k}}$. To check continuity we just need to check that the composites $s_F \circ \pi^0$, $s_F^0 \circ \pi^0$ are continuous. But $s_F \circ \pi^0(\vec{u}_1, \dots, \vec{u}_n) = \left(\mathrm{proj}(\vec{x}_i)\right)_{i=1}^n = \left(\sum_{j=1}^n \langle u_j, x_i \rangle u_j\right)_{i=1}^n$ for any o.n. n-frame $\{u_j\}_{j=1}^n$, which is clearly cont. in the $\vec{u}_j$, and $s_F^0 \circ \pi^0 = GS \circ s_F \circ \pi^0$. $\square$

To complete the proof that the Stiefel mflds are principal bdles over $Gr_n\mathbb{R}^{n+k}$, we need to check condition (2) in the Lemma.

Claim: The map $V_n\mathbb{R}^{n+k} \underset{Gr_n\mathbb{R}^{n+k}}{\times} V_n\mathbb{R}^{n+k} \longrightarrow GL_n\mathbb{R}$

$[V] \ , \ [W] \longmapsto [A]$ s.t. $[V][A]=[W]$

(and the corresponding map for o.n. frames) are continuous.

Proof: When the matrix $[A]$ exists, it is given by

$[A]=[V]^*[W]$ for any left inverse $[V]^*$ to $[V]$:

$[V][A]=[W] \Rightarrow [V]^*[V][A]=[V]^*W \Rightarrow [A]=[V]^*[W]$

(Note that mult'n by the $(n+k)\times n$ matrix $[V]$ is a linear injection $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$.)

This left inverse $[V]^*$ can be chosen continuously, as was shown in the proof of local triviality for tangent bdles. $\square$

Corollary: $V_n(\mathbb{R}^{n+k}) \xrightarrow{\pi} Gr_n\mathbb{R}^{n+k} \xleftarrow{\pi^0} V_n^0(\mathbb{R}^{n+k})$ are principal bdles.

To prove universality, we need to compute the htpy gps $\pi_* V_n(\mathbb{R}^{n+k})$, $\pi_* V_n^0\mathbb{R}^{n+k}$.

Exercise: The Gram-Schmidt process defines a deformation retraction $V_n(\mathbb{R}^{n+k}) \rightsquigarrow V_n^0(\mathbb{R}^{n+k})$, so $\pi_* V_n\mathbb{R}^{n+k} \cong \pi_* V_n^0\mathbb{R}^{n+k}$.

We'll study $\pi_* V_n^0\mathbb{R}^{n+k}$ via the following decomposition.

Prop'n: The map $V_{n+1}^0\mathbb{R}^{n+k} \xrightarrow{q} V_n^0\mathbb{R}^{n+k}$, $(\vec{u}_1,\dots,\vec{u}_{n+1}) \longmapsto (\vec{u}_1,\dots,\vec{u}_n)$ is a (locally trivial) fiber bdle with fiber $S^{k-1}=V_1^0(\mathbb{R}^k)$.

[Note: in the cplx case, the fiber is $V_1^0(\mathbb{C}^k)=S^{2k-1}$.]

Pf: We'll show that this map is trivial over any open set whose closure lies inside of one of the open sets $\pi_n^{-1}\mathcal{O}_F \subseteq V_n^0\mathbb{R}^{n+k}$, where $V_n^0\mathbb{R}^{n+k} \xrightarrow{\pi} Gr_n\mathbb{R}^{n+k}$.

Since $V_n^0(\mathbb{R}^{n+k}) \subseteq (\mathbb{R}^{n+k})^n$ is a metric space, it is normal, and hence we can find a covering $\{U_i\}_{i\in I}$ such that for each i $\exists F$ with $\overline{U}_i \subseteq \pi^{-1}\mathcal{O}_F$.

We define
$$\begin{tikzcd} \pi^{-1}\mathcal{O}_F \times V_1^0((\mathrm{Span} F)^\perp) \arrow[rr, "\alpha"] \arrow[dr] & & q^{-1}(\pi^{-1}\mathcal{O}_F) \arrow[dl] \\ & \pi^{-1}\mathcal{O}_F & \end{tikzcd}$$
by

$$\alpha((\vec{u}_1, \dots, \vec{u}_n), \vec{u}_{n+1}) = GS(\vec{u}_1, \dots, \vec{u}_n, \vec{u}_{n+1}) = \left(u_1, \dots, u_n, \frac{u_{n+1} - \mathrm{proj}_{\mathrm{Span}(\{u_i\})}(u_n)}{\|u_{n+1} - \mathrm{proj}_{\mathrm{Span}\{u_i\}}(u_n)\|}\right)$$

Note that since $\vec{u}_1, \dots, \vec{u}_n$ are already o.n., GS does not affect them, and $\{\vec{u}_1, \dots, \vec{u}_{n+1}\}$ is linearly independent because $\vec{u}_{n+1} \in (\mathrm{Span}\, F)^\perp$ and $\mathrm{Span}(\vec{u}_1, \dots, \vec{u}_n) \in \mathcal{O}_F$ (so $\mathrm{Span}(\vec{u}_1, \dots, \vec{u}_n) \cap (\mathrm{Span}\, F)^\perp = \{0\}$).

Now, if $\overline{U}_i \subseteq \pi^{-1}\mathcal{O}_F$, then $\alpha : \overline{U}_i \times \underbrace{V_1^0(\mathrm{Span}\, F^\perp)}_{\cong S^{k-1}} \to q^{-1}(\overline{U}_i)$

is a continuous map from a compact space to a Hausdorff space, to show it's a homeomorphism, we just need to show it's a bijection.

Given a point $\{u_1, \dots, u_n\} \in \pi^{-1}(\mathcal{O}_F)$, we will show that α is a bijection on the fibers over $\{\vec{u}_1, \dots, \vec{u}_n\}$, i.e. we'll show that

$$\begin{aligned} V_1^0((\mathrm{Span}\, F)^\perp) &\xrightarrow{\alpha} q^{-1}(\{\vec{u}_1, \dots, \vec{u}_n\}) \\ \vec{u} &\longmapsto \frac{\vec{u} - \mathrm{proj}_{\mathrm{Span}\{\vec{u}_1, \dots, \vec{u}_n\}}(\vec{u})}{\|\vec{u} - \mathrm{proj}_{\mathrm{Span}\{\vec{u}_1, \dots, \vec{u}_n\}}(\vec{u})\|} \end{aligned}$$

is a bijection.

First, consider the linear map

$$\begin{aligned} \mathrm{Span}(F)^\perp &\xrightarrow{\tilde{\alpha}} \mathrm{Span}(\{u_1, \dots, u_n\})^\perp. \\ \vec{u} &\longmapsto \vec{u} - \mathrm{proj}_{\mathrm{Span}(\{u_1, \dots, u_n\})}(\vec{u}) \end{aligned}$$

Since $\mathrm{Span}\{u_1, \dots, u_n\} \in \Theta_F$, we know that $\tilde{\alpha}$ is an isomorphism (see p. 5).
We have $\alpha(\vec{u}) = \tilde{\alpha}(\vec{u}) / \|\tilde{\alpha}(\vec{u})\|$, so if $\alpha(\vec{u}) = \alpha(\vec{v})$, then

$$\frac{\tilde{\alpha}(u)}{\|\tilde{\alpha}(\vec{u})\|} = \frac{\tilde{\alpha}(v)}{\|\tilde{\alpha}(v)\|} \Rightarrow \tilde{\alpha}(u) = \frac{\|\tilde{\alpha}(\vec{u})\|}{\|\tilde{\alpha}(\vec{v})\|}\tilde{\alpha}(v) \underset{\boxed{\tilde{\alpha} \text{ is linear}}}{=} \tilde{\alpha}\left(\frac{\|\tilde{\alpha}\vec{u}\|}{\|\tilde{\alpha}\vec{v}\|}\vec{v}\right)$$

$$\underset{\boxed{\tilde{\alpha} \text{ injective}}}{\Longrightarrow} \vec{u} = \frac{\|\tilde{\alpha}(\vec{u})\|}{\|\tilde{\alpha}(\vec{v})\|}\vec{v}. \quad (\star)$$

Now, the domain of α is $V_1^0(\mathrm{Span}(F)^\perp)$ = unit sphere in $(\mathrm{Span}\, F)^\perp$, so $\|\vec{u}\| = \|\vec{v}\| = 1$. Equation $(\star)$ says that the unit vectors $\vec{u}$ and $\vec{v}$ are positive multiples of one another, so they must be equal. Thus α is injective.

To show that α is surjective, consider any

$$\vec{v} \in q^{-1}\pi^{-1}(\mathrm{Span}\{\vec{u}_1, \dots, \vec{u}_n\}).$$

Then by def'n of q, we know that $\{\vec{u}_1, \dots, \vec{u}_n, \vec{v}\}$ is

an orthonormal set, i.e. an element of $V_{n+1}^0(\mathbb{R}^{n+k})$.
In particular, $\vec{v} \in \mathrm{Span}\{\vec{u}_1, \dots, \vec{u}_n\}^\perp$, and $\|\vec{v}\| = 1$.
Since $\tilde{\alpha} : \mathrm{Span}(F)^\perp \longrightarrow \mathrm{Span}(\{u_1, \dots, u_n\})^\perp$ is surjective,

$\exists\, \vec{u}_{n+1} \in \mathrm{Span}(F)^\perp$ such that

$$\vec{v} = \tilde{\alpha}(\vec{u}_{n+1})$$

$$\text{Now, } \alpha\left(\frac{\vec{u}_{n+1}}{\|\vec{u}_{n+1}\|}\right) = \frac{\tilde{\alpha}\left(\pm \frac{\vec{u}_{n+1}}{\|\vec{u}_{n+1}\|}\right)}{\left\|\tilde{\alpha}\left(\frac{u_{n+1}}{\|u_{n+1}\|}\right)\right\|} \underset{\boxed{\tilde{\alpha}\text{ linear}}}{=} \frac{\cancel{\frac{1}{\|\vec{u}_{n+1}\|}}\,\tilde{\alpha}(\vec{u}_{n+1})}{\cancel{\frac{1}{\|\vec{u}_{n+1}\|}}\,\|\tilde{\alpha}(\vec{u}_{n+1})\|}$$

$$= \vec{V}/\|\vec{V}\| \underset{\boxed{\|V\|=1}}{=} \vec{V},$$

So α is surjective. $\square$

To compute htpy, we need:

<u>Theorem</u>: For any fiber bdle $\begin{array}{c} F \xrightarrow{i} E \\ \downarrow p \\ B \end{array}$ there exists

a LES $\cdots \longrightarrow \pi_* F \xrightarrow{i_*} \pi_* E \xrightarrow{p_*} \pi_* B \xrightarrow{\partial} \pi_{*-1} F \xrightarrow{i_*} \cdots \longrightarrow \pi_0 B.$

Using this fact, we can complete the proof of universality.

<u>Proof of Thm 1</u>: By the Cellular Approx'n Thm, all maps $S^m \longrightarrow S^k$ are null homotopic for $m < k-1$, i.e. $\pi_m S^k = 0$ for $m \leq k-1$.

The LES for the bdle $\begin{array}{c} S^k \longrightarrow V_n^0 \mathbb{R}^{n+k} \\ \downarrow q_n \\ V_{n-1}^0 \mathbb{R}^{n+k} \end{array}$ now has the form

$0 \longrightarrow \pi_* V_n^0(\mathbb{R}^{n+k}) \xrightarrow{(q_n)_*} \pi_* V_{n-1}^0(\mathbb{R}^{n+k}) \longrightarrow 0$ for $* \leq k-1$.