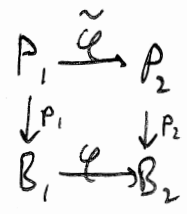


Lemma: Any map of principal G -bundles $P_1 \xrightarrow{\tilde{\varphi}} P_2$ induces an isomorphism $P_1 \xrightarrow{\cong} \varphi^* P_2$.



PF: We have a map $P_1 \rightarrow \varphi^* P_2$ which is equivariant $x \mapsto (p_1(x), \tilde{\varphi}(x))$ and covers Id_{B_1} . The result now follows from the Prop'n. \square

To prove that every bdl $\begin{array}{c} P \\ \downarrow \\ X \end{array}$ over a CW cplx is pulled back from the universal bdl $\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$, we just need to construct an equivariant map $P \xrightarrow{\tilde{\varphi}} EG$. Since $X = P/G$, the diagram $\begin{array}{ccc} P & \xrightarrow{\tilde{\varphi}} & EG \\ \downarrow & & \downarrow \\ X = P/G & \xrightarrow{\varphi} & BG = EG/G \end{array}$ can always be filled in.

Key Lemma (Ehresman): Given a principal G -bdl $\begin{array}{c} P \\ \downarrow \\ X \end{array}$ and a G -space E , there is a bijection between G -equivariant maps $P \rightarrow E$ and sections of the mixed bundle $\begin{array}{c} P \times_G E = (P \times E) / (p, e) \sim (pg, eg) \\ \downarrow \\ X \end{array}$.

Proof: Given a G -map $P \xrightarrow{\varphi} E$, we have a diagram: $\begin{array}{ccc} P & \xrightarrow{\varphi} & E \\ \downarrow & & \downarrow \\ X = P/G & \xrightarrow{S} & P \times_G E \end{array}$

Equivariance of φ implies that $\pi_0(\text{Id}, \varphi)$ factors through P/G , and S is the desired section.

In the other direction, given $\begin{matrix} P \times_G E \\ \uparrow \downarrow \pi \\ P/G = X \end{matrix}$, we define $P \xrightarrow{\varphi} E$ as follows:

We can always write $s([p]) = [p, e]$ for some (unique) $e \in E$ and we set $\varphi(p) = e$. Continuity of φ can be checked

locally, using the fact that $(U \times G) \times_G E \xrightarrow{\cong} U \times E$. \square

$$\begin{matrix} [u, g, e] \longmapsto (u, eg^{-1}) \\ [u, 1, e] \longleftarrow (u, e) \end{matrix}$$

Lecture 4:

Prop'n: Let X be a CW cplx, and let $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a fiber bundle with fiber F . If $\pi_* F = 0$ for $* \leq \dim X$ then E admits a section (if $\dim X = \infty$, we just require $\pi_* F = 0$ for $* \geq 0$).

In particular, the mixed bdlrs $\begin{matrix} P \times_G EG \\ \downarrow \\ X \end{matrix}$ always admit sections, so we obtain the required

G-map $P \rightarrow EG$.

In the proof, we will need to use the fact that every principal G -bdle over a contractible space is trivial. This follows from the Bundle Hty Theorem, which we'll prove next time.

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$$\begin{array}{ccc} [u, g, e] & \mapsto & (u, eg^{-1}) \\ Eu, 1, e & \xrightarrow{\cong} & (u, e) \end{array}$$

Lecture 4:

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In particular, the mixed bdl's $\begin{array}{c} P \times_G EG \\ \downarrow \\ X \end{array}$

always admit sections, so we obtain the required

G -map $P \rightarrow EG$.

In the proof, we will need to use the fact that every principal G -bdle over a contractible space is trivial. This follows from the Bundle Hom. Theorem, which we'll prove next time.

Proof of Prop'n: By induction on skeleta.

Certainly we can define a section $s^{(0)}: X^{(0)} \rightarrow E$. Now

assume we have a section $s^{(n)}$ defined on $X^{(n)}$; we must

extend it over each $(n+1)$ -cell. Let $\psi: D^{n+1} \rightarrow X$

be the characteristic map for an $(n+1)$ -cell in

X , or $\psi|_{S^n}: S^n \rightarrow X^{(n)}$. Then since $D^{n+1} \cong \mathbb{R}^n$,

the pullback ψ^*E is a trivial bundle over D^{n+1} .

The composite $S^n \xrightarrow{\psi} X^{(n)} \xrightarrow{s^{(n)}} E$ gives us a section

of $\sigma: \begin{matrix} \psi^*E \\ \downarrow \\ S^n \end{matrix} : \sigma(x) = (x, s^{(n)}(\psi(x)))$. Since $\psi^*E \cong \mathbb{R}^n \times F$,

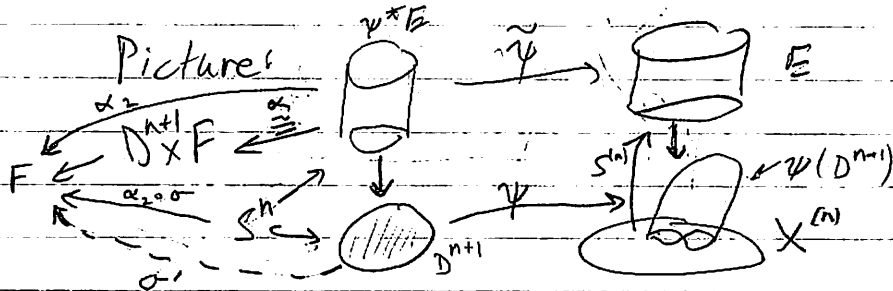
we get a corresponding map $\alpha: S^n \rightarrow F$ which extends

to $\sigma': D^{n+1} \rightarrow F$ (b/c $\pi_n F = 0$). Letting $\tilde{\sigma}: D^{n+1} \rightarrow E$ be the map

$\tilde{\sigma}(x) = \tilde{\psi}^{-1}(x, \sigma'(x))$ (where $\begin{matrix} \psi^*E & \xrightarrow{\tilde{\psi}} & E \\ \downarrow \psi & & \downarrow \\ D^{n+1} & \xrightarrow{\psi} & X \end{matrix}$) the desired ext'n of $s^{(n)}$

is: $D^{n+1} \cup_{\psi} X^{(n)} \xrightarrow{\tilde{\sigma} \cup s^{(n)}} E$ (tracing the def'n shows that

that $\tilde{\sigma}(x) = s^{(n)}(\psi(x))$ for $x \in S^n$, so this is well-defined and continuous).



A key tool for studying bundles is the following result:
The Bundle Homotopy Theorem: Let B be a paracompact Hausdorff

space, and let $\begin{array}{c} E \\ \downarrow p \\ B \times I \end{array}$ be a principal G -bdle. Then there is

a bundle isomorphism $\begin{array}{ccc} E & \cong & E_0 \times I \\ & \searrow & \swarrow \\ & B \times I & \end{array}$, where $E_0 = p^{-1}(B \times \{0\})$.
 [The corresponding result for vector bdles/Euclidean bdles follows by ditching/mixing.]

Remark: This isomorphism restricts to give isomorphisms $E_t \cong E_0$ for all $t \in I$, where $E_t = p^{-1}(B \times \{t\})$. In particular, $E_0 \cong E_1$, so we can think of the BHT as saying that "homotopic bdles" are isomorphic.

Corollary 1: If $f_0, f_1: X \rightarrow Y$ are homotopic and $\begin{array}{c} E \\ \downarrow \\ Y \end{array}$

is a principal G -bdle, then there is an isomorphism

$$\begin{array}{ccc} f_0^* E & \cong & f_1^* E \\ & \searrow & \swarrow \\ & X & \end{array}$$

PF: Any homotopy $H: X \times I \rightarrow Y$ ($H_0 = f_0, H_1 = f_1$) induces a

bundle htpy $\begin{array}{c} H^* E \\ \downarrow \\ X \times I \end{array}$ connecting $f_0^* E$ to $f_1^* E$. \square

This corollary tells us that we have a well-defined factorization

$$\begin{array}{ccc} f & \longmapsto & f^* E \\ \text{Map}(X, B) & \longrightarrow & \text{Prin}_G(X) \\ & \searrow & \swarrow \\ & [X, B] & \end{array}$$

for any paracomp. Hausd. space X and any bdle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$.

Corollary 2: If $F: X \xrightarrow{\cong} Y$ is a htpy equiv. b/w paracomp. Hausd. spaces, then F induces a bijection $F^*: \text{Prin}_G Y \rightarrow \text{Prin}_G X$.

PF: If $g: Y \rightarrow X$ is a htpy inverse to f , then $f^*g^* \cong Id_Y$

$g^*f^* \cong Id_X$. So $g^*f^*(E) \cong (Id_X)^*E = E$ and $f^*g^*E \cong (Id_Y)^*E = E. \square$

In particular, this shows that all bdlcs over a contractible (paracomp., Hausdorff) space are trivial; we used this already in the case where the base is a disk.

We can now complete the proof of our theorem on universal bdlcs:

Proof: We want to show that $Map(X, B) \xrightarrow{f} Prin_G(B)$ induces a bijection $[X, B] \xrightarrow{\cong} Prin_G(B)$

when X is CW and $\pi_* E = 0$. This map is well-defined by the BHT, [Note: every CW cplx is paracompact, by a thm of Miyazaki.] and we've proven that it's surjective.

To prove injectivity, we must show that if $f^*E \cong g^*E$ for some $f, g: X \rightarrow B$, then $f \cong g$. This is just a "relative"

form of the surjectivity statement, and we prove it the same way:

letting $P = f^*E$, we have diagrams $\begin{matrix} P & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B \end{matrix}$, $\begin{matrix} P & \xrightarrow{\tilde{g}} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & B \end{matrix}$ which (by the Key Lemma) correspond to sections s_f, s_g of $P^*_G E$.

These give us a partially defined section of $P_X^G(E \times I)$ (defined on $X \times \{0, 1\}$) and just as before we can extend this section over the rest of the CW cplx $X \times I$. This

gives a section $s: X \times I \rightarrow P_X^G(E \times I)$ which translates under $(x, t) \mapsto [x, e, t]$

the Key Lemma to a map $F: X \times I \rightarrow B \times I$, and the composite $X \times I \xrightarrow{F} B \times I \rightarrow B$ is the desired htpy bldg $f, g: X \rightarrow B$. \square

Proposition: If $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is a principal G -bldg w/ $\pi_* E = 0$ for $* \leq n$,

then $[X, B] \xrightarrow{F} \text{Prin}_G(X)$ is bijective for CW cplx's X of $\dim^n \leq n$, surjective for $\dim X = n$.

(We call E n -universal.)

Pf: Just follow the previous proof, noting that $\dim(X \times I) = n+1$,

so we don't get injectivity if $\pi_{n+1} E \neq 0$. \square

To prove the BHT, we need two lemmas:

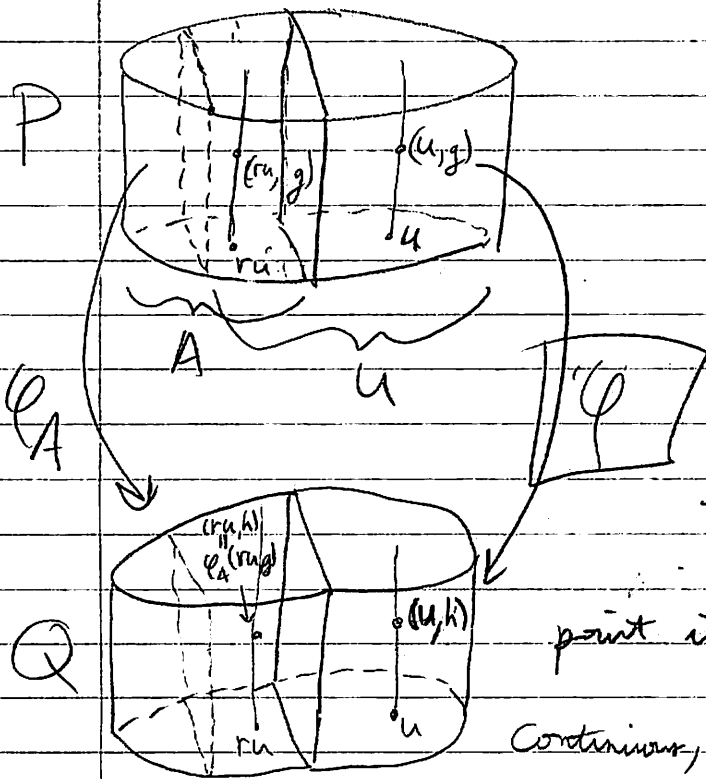
Lemma 1:

Consider a subset $A \in X$. If $\begin{matrix} P & Q \\ \downarrow & \downarrow \\ X & X \end{matrix}$ are principal G -bldgs over X , which are both trivial over some open set $U = \overline{X - A}$ that retracts to $U \cap A$. ← "closure"

Then any isomorphism $\begin{matrix} P|_A \xrightarrow{\cong} Q|_A \\ \downarrow \cong \downarrow \\ A \end{matrix}$ extends to an isomorphism $\begin{matrix} P \xrightarrow{\cong} Q \\ \downarrow \cong \downarrow \\ X \end{matrix}$.

Pf: To simplify notation, we'll identify $P|_U$ and $Q|_U$ with $U \times G$.

We extend φ_A using the diagram



So we define

$$\varphi(u, g) = (u, \pi_2(\varphi_A(ru, g)))$$

$$\text{where } \pi_2: U \times G \rightarrow G.$$

$$\parallel$$

$$Q|_U$$

This agrees with φ_A on any

point in $Q|_A$. To see that φ is

continuous, just note that since $U \supset \overline{X-A}$,

we have $X = \text{int}(A) \cup U$, and $P = P|_{\text{int}(A)} \cup P|_U$.

So we have defined φ on two open sets, and our definitions

agree on the overlap. \square

Lemma 2: For any principal G -bdle $\begin{matrix} P \\ \downarrow \\ B \times I \end{matrix}$ and any

pt. $b \in B$, \exists an open nbhd $U_b \ni b$ s.t. $P|_{U_b \times I} \cong U_b \times I \times G$.

Pf: For each $t \in I$, \exists an nbhd $U_{b,t} \times (t - \epsilon_t, t + \epsilon_t)$ of $B \times I$ over

which P is trivial. By compactness of I , $\exists t_1, \dots, t_n \in I$ s.t.

$I = \bigcup_{i=1}^n (t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i})$. We set $U_b = \bigcap_{i=1}^n U_{b, t_i}$ so that

P is trivial over $U_b \times [0, s_1]$, $U_b \times (s_1 - \varepsilon, s_2]$, ..., $U_b \times (s_{n-1} - \varepsilon, 1]$
 for some $0 < s_1 < \dots < s_n$ (the s_i are easily chosen in terms of
 the t_i). Now Lemma 1 can be applied $n-1$ times,
 h.c. $U_b \times (s_i - \varepsilon, s_{i+1}]$ retracts to $U_b \times (s_i - \varepsilon, s_i]$. \square

Proof of the BHT: Given $\begin{matrix} P \\ \downarrow \\ B \times I \end{matrix}$, we need to

produce an isom. $P \cong P_0 \times I$. Let $\{U_b\}_{b \in B}$ be the
 cover guaranteed by Lemma 2, or $P|_{U_b \times I} \cong U_b \times I \times G$.

Since B is paracompact and Hausdorff, we can find a locally finite

open refinement $\{V_i\}_{i \in I}$ of $\{U_b\}_{b \in B}$ and a partition

of unity $\{p_i\}_{i \in I}$ subordinate to $\{V_i\}$ (i.e. $\sum_{i \in I} p_i(b) = 1$

for each $b \in B$, and $\text{supp}(p_i) = \{b \in B \mid p_i(b) > 0\} \subseteq V_i$).

If we choose a well-ordering of I , then we can define $W_0 = B \times \{0\}$,

and $W_i = \{(b, t) \mid t \leq \sum_{j \leq i} p_j(b)\}$ (for each $i \in I$).

We now proceed by transfinite induction over I :

each U_b

For each $i \in I$, consider the statement:

$S(i)$: Given any sequence of compatible isom's $P|_{W_j} \xrightarrow{\cong} P_0 \times I|_{W_j}$ for $j < i$, there exists an isom. $P|_{W_i} \xrightarrow{\cong} P_0 \times I|_{W_i}$ s.t. $\varphi_i|_{W_j} = \varphi_j$ for $j < i$. Moreover, such a comp. seq. exist.

We prove that $S(i)$ holds for all $i \in I$ by transfin. induction.

$S(0)$ is trivially true. Assuming $S(j)$ for all $j < i$,

we must prove $S(i)$.

We claim that there is an isom.

$P|_{\bigcup_{j < i} W_j} \xrightarrow{\cong} P_0 \times I|_{\bigcup_{j < i} W_j}$. To define φ , we use the axiom of

choice: for every collection of comp. isoms $\{F_A: P|_A \cong P_0 \times I|_A\}_{A \in \mathcal{A}}$

(where \mathcal{A} is any collection of subsets of $B \times I$) for which there exists

an extn $P|_{\bigcup_{A \in \mathcal{A}} A} \cong P_0 \times I|_{\bigcup_{A \in \mathcal{A}} A}$, we choose such an extn.

Then by our induction hypothesis, we have chosen isom's

$P|_{W_j} \xrightarrow{\cong} P_0 \times I|_{W_j}$, $j < i$ which restrict to the identity on $W_0 = B \times \{0\}$

and satisfy $\varphi_l|_{W_k} = \varphi_k$ for $k < l$. Now $P|_{\bigcup_{j < i} W_j} \xrightarrow{\cong} P_0 \times I|_{\bigcup_{j < i} W_j}$.

Using Lemma 1, we will extend $\varphi = \bigcup_{j < i} \varphi_j$ to all of W_i .

We have a retraction $W_i \rightarrow \bigcup_{j < i} W_j$
 $(b, t) \mapsto (b, \min(t, \sum_{j < i} p_j(b)))$

In addition,

$$W_i - \bigcup_{j < i} W_j = \{(b, t) \mid \sum_{j < i} p_j(b) < t \leq \sum_{j \leq i} p_j(b)\} \subseteq (\text{Supp}(p_i) \times I) \cap W_i \\ \subseteq (V_i \times I) \cap W_i.$$

The set $(V_i \times I) \cap W_i$ is an open subset of W_i on which both $P_0 \times I$ and P are trivial, and it contains

$\overline{W_i - \bigcup_{j < i} W_j}$ b/c it contains the closed set $(\text{Supp}(p_i) \times I) \cap W_i$.

By Lemma 2, we can extend the isom. $P_0 \times I|_{\bigcup_{j < i} W_j} \xrightarrow{\cong} P|_{\bigcup_{j < i} W_j}$ to an isom. $P_0 \times I|_{W_i} \cong P|_{W_i}$

and moreover we can do the same for any compatible

family of isoms $\varphi_j: P|_{W_j} \xrightarrow{\cong} P_0 \times I|_{W_j}$ ($j < i$). \square

Remark: Milnor shows, by point-set arguments,

that transfinite induction can be avoided: by Lemma 5.9

in MS, there exists a countable cover $\{U_i\}_{i=1}^{\infty}$ of $B \times I$ over

which P is trivial. The process in Lemma 1 then produces

a new cover of B consisting of finite intersections of U_i 's;

hence this new cover is still countable. [Note: one needs

here the easy fact that $B \text{ paracompact} \Rightarrow B \times I \text{ paracompact.}$]

Lecture 5

Characteristic Classes:

Defn: A characteristic class for principal G -bundles is a natural transformation

$$\text{Prin}_G(X) \xrightarrow{c} H^n(X; A),$$

defined for all CW complexes X (n and A are fixed; c are the dim'n and coeff. gp for the class).

Naturality just means that given a map $f: X \rightarrow Y$

and a bdl $\begin{array}{c} P \\ \downarrow \\ Y \end{array}$, we have $c(f^*Y) = f^*(c(P)) \in H^n(X; A)$.

Lemma (Yoneda):

Say $\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$ is a universal principal G -bdle w/ B a CW cplx.

Then there is a bijection

$$\left\{ \begin{array}{l} \text{Char. classes for } G\text{-bdlcs} \\ \text{w/ coeff. gp. } A \text{ and dim'n } n \end{array} \right\} \xrightarrow{\cong} H^n(BG; A).$$

PF: The bijection is given by $c \mapsto c(EG) \in H^n(BG; A)$.

It remains only to note that for any $\alpha \in H^n(BG; A)$,

the assignment $P \mapsto F^*(\alpha)$ (where F is a classifying map for P) defines a characteristic class; this uses

the bijection $\text{Prin}_G(X) \cong [X, BG]$. □