Lecture 2

Basic Examples: Spheres and Projective Spaces

Def'n: $S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid \Sigma x_i^2 = 1\}$ (the unit sphere).

Fact: S^{n-1} is a smooth mfl'd.

Proof: We can cover S^{n-1} by local parametrizations arising from projection onto hyperplanes:

$$\varphi_i^{\pm} : \underbrace{D^{n-1}}_{=\{\vec{x} \in \mathbb{R}^{n-1} \mid \Sigma x_j^2 < 1\}} \longrightarrow S^{n-1}$$

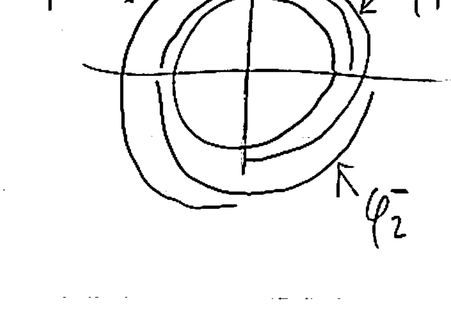
$$(x_1, \ldots, x_{n-1}) \longmapsto (x_1, \ldots, \underset{i^{th}\text{ position}}{\pm\sqrt{1-\Sigma x_j^2}}, \ldots, x_{n-1}).$$

This gives $2n$ local parametrizations, since we can choose any $i \in \{1, \ldots, n\}$ and either sign $+/-$.

These maps are homeomorphisms onto their images, because they are inverse to the relevant projections.

The transition fcns $(\varphi_j^{\pm})^{-1}\varphi_i^{\pm}$ are then clearly smooth (since $\sqrt{y}$ is smooth away from $y = 0$). $\square$

Example: $S^1 \subset \mathbb{R}^2$ is covered by 4 charts:

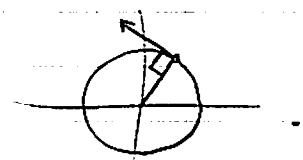


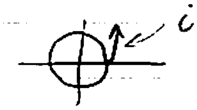
<u>Prop'n</u>: The tangent bundle to S^1 is trivial.

<u>Proof</u>: We have a continuous section $S^1 \to T(S^1)$.
$(x_1, x_2) \mapsto (x_2, -x_1)$

One can check (using MS Lemma 1.2) that $(x_2, -x_1) \in T_{(x_1,x_2)} S^1$. $\square$

Geometrically, this function just rotates (x_1, x_2) counter-clockwise by $90°$:



This can be obtained more systematically from complex multiplication in $\mathbb{R}^2 \cong \mathbb{C}$: starting with the tangent vector $(0,1) = i$ in $T_{(1,0)} S^1$ ()

we simply multiply by $e^{i\theta} = (\cos\theta, \sin\theta)$ to obtain a tangent vector at $e^{i\theta} = (\cos\theta, \sin\theta) \in \mathbb{C} = \mathbb{R}^2$:

$$\begin{aligned} ie^{i\theta} = i(\cos\theta + i\sin\theta) &= i\cos\theta - \sin\theta \\ &= (-\sin\theta, \cos\theta). \end{aligned}$$

So this process transports the vector $(0,1) \in T_{(1,0)} S^1$ to the vector $(-\sin\theta, \cos\theta) \in T_{(\cos\theta, \sin\theta)} S^1$. Letting $x_1 = \cos\theta$, $x_2 = \sin\theta$ gives the original formula.

This process really shows that any Lie group (= smooth mfld equipped with smooth mult'n + inverse maps) has trivial tangent bundle. (See MS p.20 for the case of S^3.)

which is the group of unit quaternions).

Def'n (Real Projective Space)

$$\mathbb{R}P^n = S^n \Big/ \substack{\vec{x} \sim -\vec{x} \\ \text{for all } \vec{x} \in S^n}.$$

We equip $\mathbb{R}P^n$ with the quotient topology.

Fact: $\mathbb{R}P^n$ is a smooth mfld.

Proof: We can use the same charts as for S^n, except that we must compose with the projection $\pi: S^n \to \mathbb{R}P^n$. Note that $\varphi_i^{\pm}(D^n)$ never contains two antipodal points, so $\pi \circ \varphi_i^{\pm}: D^n \to \mathbb{R}P^n$ is still a homeomorphism onto its image. The transition maps are exactly the same as those for the sphere. $\square$

The canonical line bdle over $\mathbb{R}P^n$:

We have a line bdle γ_n^1 over $\mathbb{R}P^n$, whose fiber over $\{\vec{x}, -\vec{x}\} \in \mathbb{R}P^n$ consists of the line $\{c\vec{x} \mid c \in \mathbb{R}\} \subseteq \mathbb{R}^{n+1}$. Formally, the total space of γ_n^1 is given by

$$E(\gamma_n^1) = \left\{ (\{\pm\vec{x}\}, \vec{v}) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \,\middle|\, \vec{v} = c\vec{x} \text{ for some } c \in \mathbb{R} \right\},$$

with the subspace topology. The projection is $\begin{aligned} E(\gamma_n^1) &\to \mathbb{R}P^n \\ (\{\pm x\}, v) &\mapsto \{\pm x\} \end{aligned}$

and each fiber has its natural vector space structure. It is easy to check (MS p. 16) that γ_n^1 is locally trivial (over the above coord nbhds for $\mathbb{R}P^n$, say).

<u>Theorem 2.1</u> γ_n^1 is not trivial, for any $n \geq 1$.

<u>Proof</u>: Trivial bundles $\begin{matrix} V \\ \pi\downarrow \\ X \end{matrix}$ admit nowhere-zero sections $\begin{matrix} V \\ s\uparrow\downarrow\pi \\ X \end{matrix}$ (with $\pi s = id_X$). If γ_n^1 had such a section s, then

$$S^n \xrightarrow{\pi} P^n \xrightarrow{s} E(\gamma_n^1)$$
$$x \longmapsto \{\pm x\} \longmapsto (\{\pm x\}, c(x)\cdot x)$$

gives a continuous function $c: S^n \longrightarrow \mathbb{R}-\{0\}$ with $c(x) = -c(-x)$, but this contradicts the Intermediate Value Theorem. $\square$

<u>Rmk</u>: Continuity of c, and the fact that the obvious local trivializations of γ_n^1 are homeomorphisms, are essentially the same and can be proven just as in the proof of local triviality for tangent bdles.

Clutching Functions

Vector bundles can be built up from trivial bundles $U \times \mathbb{R}^n$ via clutching functions:

Def'n: If $V \xrightarrow{\pi} B$ is a vector bundle and $\varphi_1 : \pi^{-1} U_1 \xrightarrow{\cong} U_1 \times \mathbb{R}^n$, $\varphi_2 : \pi^{-1} U_2 \xrightarrow{\cong} U_2 \times \mathbb{R}^n$ are trivializations with $U_1 \cap U_2 \neq \emptyset$, then the isomorphism

$$\varphi_{21} := \varphi_2 \circ \varphi_1^{-1} : U_1 \cap U_2 \times \mathbb{R}^n \xrightarrow{\cong} U_1 \cap U_2 \times \mathbb{R}^n$$

is called a transition fcn for V. For each $x \in U_1 \cap U_2$, $\varphi_{21} : \{x\} \times \mathbb{R}^n \longrightarrow \{x\} \times \mathbb{R}^n$ is a linear isomorphism, so we may view φ_{21} as a mapping $U_1 \cap U_2 \longrightarrow GL_n(\mathbb{R})$, called a clutching function.

Lemma (the cocycle condition):

If $U_1 \cap U_2 \cap U_3 \neq \emptyset$ and $\varphi_i : \pi^{-1} U_i \xrightarrow{\cong} U_i \times \mathbb{R}^n$, then we have

$$\boxed{\varphi_{32} \varphi_{21} = \varphi_{31}}$$ (either as comp. of fcns or as product of matrices)

Proof: $\varphi_{32} \circ \varphi_{21} = (\varphi_3 \varphi_2^{-1})(\varphi_2 \circ \varphi_1^{-1}) = \varphi_3 \circ \varphi_1^{-1} = \varphi_{31}$. □

We can now construct v. bdles using clutching fcns:

Prop'n: If $B = \bigcup_{i \in I} U_i$ (open sets) and $\varphi_{ji} : U_i \cap U_j \rightarrow GL_n \mathbb{R}$ are clutching fcns satisfying the cocycle condition, then

$$V = \left(\coprod_{i \in I} U_i \times \mathbb{R}^n \right) \Big/ (u, \vec{v}) \sim (u, \varphi_{ji} \vec{v}) \text{ for all } u \in U_i,\ i \in I$$

is a v. bdle over B w/ clutching fcns φ_{ji}.

Principal Bundles

When we have a vector bdle described in terms of clutching fcns, we can get rid of $\mathbb{R}^n$ entirely:

Say $\begin{matrix} V \\ \downarrow \\ B \end{matrix}$ has clutching fcns $\varphi_{ji} : U_i \cap U_j \to GL_n\mathbb{R}$. ($\{U_i\}$ an open cover of B)

Then we can use these fcns to construct a bundle whose ***fibers*** are $GL_n\mathbb{R}$ itself:

$$P = P_V = \left(\coprod_i U_i \times GL_n\mathbb{R} \right) \Big/ \sim$$

where $\underset{\substack{\wedge \\ U_i \times GL_n\mathbb{R}}}{(u, A)} \sim \underset{\substack{\wedge \\ U_j \times GL_n\mathbb{R}}}{(u, \varphi_{ji}(u)A)}$ if $u \in U_i \cap U_j$, $A \in GL_n\mathbb{R}$.

[We only glue pts from different elements of the <u>disjoint union</u>.]

The projections $U_i \times GL_n\mathbb{R} \to U_i \hookrightarrow B$ respect the equivalence reln, and yield a continuous projection map $P \xrightarrow{\pi} B$. Each fiber of this map is <u>non</u>-<u>canonically</u> homeomorphic to $GL_n\mathbb{R}$, and in fact $\pi^{-1}(U_i) \cong U_i \times GL_n\mathbb{R}$ for each i.

We can recover V by <u>mixing</u>: the space V admits a well-defined <u>right</u> action of $GL_n\mathbb{R}$, given by

$$(u, A) \cdot B = (u, AB).$$

<u>Lemma</u>: $P_V \times_{GL_n\mathbb{R}} \mathbb{R}^n \cong V$ as vector bdles over B.

Here $P_V \times_{GL_n\mathbb{R}} \mathbb{R}^n := (P_V \times \mathbb{R}^n) \Big/ \begin{pmatrix} (p,x) \sim (pA, A^{-1}x) \\ \text{for all } p \in P,\ x \in \mathbb{R}^n,\ A \in GL_n\mathbb{R} \end{pmatrix}$.

<u>Proof</u>: We have a map

$$\begin{aligned} P \times \mathbb{R}^n &\longrightarrow V \quad \left(= \left(\coprod_i U_i \times \mathbb{R}^n\right)\Big/\sim \right) \\ ([U_i, A], x) &\longmapsto [U_i, Ax] \end{aligned}$$

which is continuous and factors through the quotient on the left. On each fiber, it can be identified with the linear isomorphism $x \mapsto Ax$, and its inverse is given by the continuous map

$$\begin{aligned} P \times_{GL_n\mathbb{R}} \mathbb{R}^n &\longleftarrow \coprod_i U_i \times \mathbb{R}^n \\ [[U_i, I], x] &\longleftarrow (U_i, x) \end{aligned}$$

which factors through the equivalence rel'n on the right. $\square$

Note that the vector space structure on $(P \times_{GL_n\mathbb{R}} \mathbb{R}^n)|_b$ is just that inherited from $\mathbb{R}^n$.

These constructions allow us to pass between vector bundles and principal $GL_n\mathbb{R}$-bundles. To make the correspondence complete, we need a notion of <u>maps</u> b/w principal $GL_n\mathbb{R}$ bundles:

If $\begin{matrix} P_1 \\ \downarrow \pi_1 \\ B_1 \end{matrix}$ and $\begin{matrix} P_2 \\ \downarrow \pi_2 \\ B_2 \end{matrix}$ are principal $GL_n\mathbb{R}$-bundles,

then a <u>map</u> $P_1 \to P_2$ consists of a commutative diagram

$$\begin{tikzcd} P_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi_1"] & P_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "\varphi"] & B_2 \end{tikzcd}$$

in which $\tilde{\varphi}$ is <u>equivariant</u>: $GL_n\mathbb{R}$ acts on both P_1 and P_2, and $\tilde{\varphi}$ must satisfy

$$\tilde{\varphi}(p \cdot A) = \tilde{\varphi}(p) \cdot A$$

for all $p \in P_1$ and $A \in GL_n\mathbb{R}$.

<u>Exercise</u>: Maps b/w vector bdles induce maps b/w the associated $GL_n\mathbb{R}$-bundles, and vice-versa, and these correspondences respect composition of maps. Hence in particular, when applied to the identity map $V \xrightarrow{id_V} V$ we see that different trivializations produce the same $GL_n\mathbb{R}$ bundle up to isomorphism.

Metrics and Principal $O(n)$-bundles:

Recall that an inner product $\langle , \rangle$ on a real v.sp. V is a symmetric, bilinear function $V \times V \longrightarrow \mathbb{R}$, $v, w \longmapsto \langle v, w \rangle$ which is positive definite, i.e. $\langle v, v \rangle > 0$ for $v \neq 0$.

We want to consider vector bundles $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ equipped with a metric on each fiber.

Def'n: A Euclidean vector bundle is a real v. bdle $\begin{smallmatrix} E \\ \downarrow \pi \\ B \end{smallmatrix}$ together w/ a continuous function

$$\underbrace{E \times_B E}_{= \{(v,w) \in E \times E \mid \pi(v) = \pi(w)\}} \xrightarrow{\langle , \rangle} \mathbb{R}$$

whose restriction to each subset $\pi^{-1}(b) \times \pi^{-1}(b) \subseteq E \times_B E$ $(b \in B)$ is an inner product on $\pi^{-1}(b)$.

Alternate Viewpoint: An inner product on a real v. space V is equivalent to a positive definite quadratic function $\mu : V \to \mathbb{R}$. This means $\mu(v) > 0$ for $v \neq 0$, and $\mu(v) = \sum l_i(v) l_i'(v)$ for some linear functions $l_i, l_i' : V \to \mathbb{R}$.

Given μ, we obtain an inner product by the "polarization" formula

$$\langle v, w \rangle = \tfrac{1}{2}\left(\mu(v+w) - \mu(v) - \mu(w)\right)$$

and given $\langle , \rangle$ we obtain the associated μ by setting

$$\mu(v) = \langle v, v \rangle =: |v|^2$$

Note that if we express v in an o.n. ("orthonormal") basis $\{e_i\}$, then

$$\mu(v) = \langle \Sigma \lambda_i e_i, \Sigma \lambda_i e_i \rangle = \sum_{i,j} \lambda_i \lambda_j,$$

and the functions $v \mapsto \lambda_i$ are all linear.

Now continuity of $\langle , \rangle : E \times_B E \to \mathbb{R}$ is equivalent to continuity of the associated $\mu : E \to \mathbb{R}$.

<u>Lecture 3</u> A Euclidean bundle is not only locally isomorphic to a trivial bundle, it is also automatically <u>isometric</u> to a trivial bundle with its standard inner product. This is Lemma 2.4 in MS, and follows from continuity of the Gram-Schmidt orthogonalization process.

<u>Clutching Functions for Euclidean Bundles:</u>

<u>Proposition:</u> Let $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ be a Euclidean v. bdle. Then there exist local trivializations $\varphi_i : U_i \times \mathbb{R}^n \to E$ such that the associated clutching functions $\varphi_{ij} : U_i \cap U_j \to GL_n\mathbb{R}$ all land inside $O(n) = \{A \in GL_n\mathbb{R} \mid AA^T = I_n\}$.

Proof: Simply choose φ_i to be the local isometries guaranteed by Lemma 2.4 (MS). $\square$

This proposition allows us to associate an $O(n)$-bdle to each Euclidean v. bdle, just as we associated a $GL_n\mathbb{R}$-bdle to each ordinary v. bdle. Again, this bdle depends only on the isometry type of the Euclidean bdle.

In the other direction, the mixed bdle

$$\begin{array}{c} P \times_{O(n)} \mathbb{R}^n \\ \downarrow \\ B \end{array} \quad \text{associated to an } O(n)\text{-bdle } \begin{array}{c} P \\ \downarrow \\ B \end{array}$$

inherits a metric from $\mathbb{R}^n$, because the transition fcns are isometries.

Remark: All of this works equally well with $\mathbb{R}$ replaced by $\mathbb{C}$. Metrics on complex bundles are required to be Hermitian, that is they are conjugate-linear in the 2^{nd} coord: $\langle v, zw\rangle = \bar{z}\langle v, w\rangle$ for $z \in \mathbb{C}$.

The transition fcns then lie in $U(n) = \{A \in GL_n\mathbb{C} \mid A\bar{A}^T = I_n\}$, so cplx Hermitian bdles correspond to (principal) $U(n)$-bundles.

Principal Bundles and their Homotopy Theory:

The $GL_n\mathbb{R}/O(n)$ bdles we have associated to vector / Euclidean bdles are examples of the general notion of principal bundles:

Def'n Let G be a topological group (i.e. the mult'n map $G\times G \to G$ and inversion $G \to G$ are continuous). A (right) principal G-bdle over a space B is a map

$$\begin{array}{c} P \\ \downarrow{\scriptstyle p} \\ B \end{array}$$

together with an open covering $\{U_i\}_{i\in I}$ of B with the following properties:

1) (Local triviality) For each $i \in I$, there is a homeomorphism

$$\varphi_i : U_i \times G \xrightarrow{\cong} p^{-1}(U_i)$$

satisfying $p\varphi_i(u,g) = u$;

2) (Principality) Whenever $U_i \cap U_j \neq \emptyset$, the composite

$$\varphi_{ji} = \varphi_j^{-1}\varphi_i : U_i \cap U_j \times G \longrightarrow U_i \cap U_j \times G$$

has the form $\varphi_{ji}(u,g) = (u, \tau_{ji}(u)\cdot g)$ for some $\tau_{ji}(u) \in G$ [here $\tau_{ji}(u)$ depends only on u, not on g].

Remark: The function $u \mapsto \tau_{ji}(u)$ is automatically continuous, b/c $\tau_{ji}(u) = \pi_2(\varphi_{ji}(u,g))\cdot(\pi_2(u,g))^{-1}$ (where $\pi_2 : U_i \cap U_j \to G$ is projection onto G).

The reason for calling this a <u>right</u> principal bdle is:

<u>Lemma</u>: If $\begin{smallmatrix} P \\ \downarrow p \\ B \end{smallmatrix}$ is a (right) principal G-bdle, then P admits a continuous <u>right</u> action $P \times G \to P$ such that

1) The quotient P/G is homeomorphic to B
2) The trivializations $\varphi_i : U_i \times G \to p^{-1}(U_i)$ are G-equivariant (where $(u,g)\cdot h := (u, gh)$).

[Note that 2) implies that G acts freely, and acts transitively on each fiber $p^{-1}(b)$.]

<u>Proof</u>: We transport the action
$$\begin{aligned}(U_i \times G) \times G &\longrightarrow U_i \times G \\ (u_i, g), h &\longmapsto u_i, gh\end{aligned}$$
to P using the local trivalizations $\varphi_i : U_i \times G \xrightarrow{\cong} p^{-1}(U_i)$:

For $\in P$, $g \in G$ we define
$$x \cdot g = \varphi_i(\varphi_i^{-1}(x) \cdot g).$$

This is well-defined by principality: if $u = p(x) \in U_i \cap U_j$, we must check that $\varphi_i(\varphi_i^{-1}(x) \cdot g) = \varphi_j(\varphi_j^{-1}(x) \cdot g)$, i.e. that $\varphi_j^{-1} \varphi_i(\varphi_i^{-1}(x) \cdot g) = \varphi_j^{-1}(x) \cdot g$.

Letting $\varphi_i^{-1}(x) = (u, h)$, we have
$$\begin{aligned}
\varphi_j^{-1}\varphi_i(\varphi_i^{-1}(x) \cdot g) &= \varphi_{ji}((u,h)\cdot g) = \varphi_{ji}(u, hg) \\
&= (u, \tau_i(u) hg) = (u, \tau_i(u) h) \cdot g \\
&= \varphi_{ji}(u,h) \cdot g = \varphi_j^{-1}(\varphi_i(u,h)) \cdot g \\
&= \varphi_j^{-1}(x) \cdot g.
\end{aligned}$$

To see that $P/G \cong B$, note that we have a comm diagram

$$\begin{tikzcd} & P \arrow[dl, "q"'] \arrow[d, "p"] \\ P/G \arrow[r, dashed, "f"] & B \end{tikzcd}$$

in which f is a continuous bijection. To see that f is an open map, consider any open set $\bar{V} \subseteq P/G$. Then $V = q^{-1}(\bar{V})$ is open in P, and $f(\bar{V}) = p(q^{-1}\bar{V})$. But p is an open map, b/c locally it is just the projection $U_i \times G \to U_i$. $\square$

<u>Basic Examples</u>:

The $GL_n\mathbb{R}/O(n)$-bdle associated to a vector/Euclidean bdle are principal bundles. In fact, for any group G and any clutching data $\varphi_{ji}: U_i \cap U_j \to G$ ($\{U_i\}_i$ an open cover of some base B), the bundle

$$\begin{array}{c} P = \left(\coprod_i U_i \times G\right) \Big/ (u,g) \sim (u, \varphi_{ji}(u)g) \\ \downarrow p \\ B \end{array}$$

is principal, w/ local trivializations the inclusions $U_i \times G \hookrightarrow P$. The associated action is just $[u,g] \cdot h = [u, gh]$.

[The fact that $U_i \times G \hookrightarrow P$ is a homeomorphism onto $p^{-1}(U_i)$ follows from the fact that $\begin{array}{c} U_i \cap U_j \times G \longrightarrow U_i \cap U_j \times G \\ u \quad g \longmapsto (u, \varphi_{ji}(u)g) \end{array}$ is a homeomorphism.]

<u>Note</u>: The fact that $U_i \times G \hookrightarrow P$ is a homeomorphism onto its image $p^{-1}(U_i)$ follows from the fact that

$$\begin{array}{ccc} U_i \cap U_j \times G & \longrightarrow & U_i \cap U_j \times G \\ (u, g) & \longmapsto & (u, \varphi_{ji}(u) g) \end{array} \quad \text{is a homeomorphism.}$$

Maps b/w Principal Bdles:

<u>Def'n</u>: If $\begin{array}{c} P_1 \\ \downarrow p_1 \\ B_1 \end{array}$ and $\begin{array}{c} P_2 \\ \downarrow p_2 \\ B_2 \end{array}$ are principal G-bdles, a map from $P_1 \to P_2$ is a diagram

$$\begin{tikzcd} P_1 \arrow[r, "\varphi"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowright"] & P_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "\bar{\varphi}"] & B_2 \end{tikzcd}$$

indicates commutativity

<u>in which φ is</u> G-equivariant.

We've seen that fiber-wise isomorphisms of V. bdles are honest isomorphisms; here is the analogue for principal bdles.

<u>Prop'n</u>: If $\begin{tikzcd} P_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & P_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$ is a map of principal G-bdles (covering Id_B) then φ is a homeomorphism (and its inverse $\varphi^{-1}: P_2 \to P_1$ is also a map of principal bdles).

<u>PF</u>: Locally, φ has the form $\begin{array}{ccc} U \times G & \xrightarrow{\varphi} & U \times G \\ (u, g) & \longmapsto & (u, \varphi_2(u,g)) \end{array}$, where $g \mapsto \varphi_2(u,g)$ is G-equivariant (wrt right mult. in G). This means $\varphi_2(u,g) = h(u) g$

where $h(u) := \varphi_2(u,g)g^{-1}$. Hence $h : U \to G$ is continuous, and now $\varphi^{-1} : U \times G \to U \times G$ is the continuous map $(u,g) \mapsto (u, h(u)^{-1}g)$.

So φ is a continuous bijection, and its inverse is continuous. $\square$

Corollary: If $\begin{matrix} P \\ \downarrow p \\ B \end{matrix}$ is a principal G-bdle admitting a continuous section $\begin{matrix} P \\ p\downarrow\uparrow s \\ B \end{matrix}$ $(ps = \mathrm{Id}_B)$ then P is trivial, i.e. there is an isom. of G-bdles
$$\begin{tikzcd} P \arrow[rr, "\cong"] \arrow[dr] & & B \times G \arrow[dl] \\ & B & \end{tikzcd}$$

Pf: Define $\varphi : \begin{matrix} B \times G \to P \\ (b,g) \mapsto s(b)\cdot g \end{matrix}$, and apply the Prop'n. $\square$

Here is another application of the Prop'n:

Exercise: Say $\begin{matrix} V \\ \downarrow \\ B \end{matrix}$ is a v. bdle, and say $\{U_i, \varphi_i\}_i$, $\{V_j, \psi_j\}_j$ are two different local trivializations of V. Then the associated principal $GL_n\mathbb{R}$ bdles for these different clutching data are isomorphic.

Pullbacks:

Given a map $F : X \to Y$ and a bdle (v. bdle, Euclidean bdle, principal G-bdle) $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$, the pullback $F^*E = \{(x,e) \in X \times E \mid F(x) = \pi(e)\}$ is a bdle over B (of the same type).

[If $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ is trivial over $\{U_i\}$, f^*E will be trivial over $\{f^{-1}(U_i)\}$.]

Our next goal is the following theorem, which describes the set $\mathrm{Prin}_G(X) = \{\text{principal } G\text{-bdles over } X\}/\cong$ of isom. classes of G-bdles <u>homotopically</u>.

<u>Theorem</u>: If $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ is a principal G-~~bdle~~ such that all htpy groups $\pi_*(E)$ are trivial, then for every CW cplx X, the map

$$\begin{array}{ccc} \mathrm{Map}(X, B) & \xrightarrow{\Phi} & \mathrm{Prin}_G(X) \\ f: X \to B & \longmapsto & [f^*(E)] \end{array}$$

factors through homotopy classes and gives a <u>bijection</u>

$$[X, B] \xrightarrow{\cong} \mathrm{Prin}_G(X).$$

<u>Notation / Terminology</u>: The bdle $E \to B$ is called a <u>universal</u> principal G-bdle. One often denotes the base space B by BG and the total space E by EG; BG is called a <u>classifying space</u> for G.

The proof of this theorem will require several important ideas, constructions and results.

We begin by considering surjectivity of Φ.