<u>Examples:</u> - $Gr_1(\mathbb{R}^n) \cong \mathbb{R}P^{n-1}$, and the universal bdle

$\gamma^1(\mathbb{R}^n) \downarrow Gr_1(\mathbb{R}^n)$ is the universal bdle over $\mathbb{R}P^{n-1}$.

- There are canonical homeomorphisms $Gr_n\mathbb{R}^{n+k} \xrightarrow[c]{\cong} Gr_k\mathbb{R}^{n+k}$, given by sending $V \subseteq \mathbb{R}^{n+k}$ to $V^{\perp} = \{w \mid \langle w, v\rangle = 0 \text{ for all } v \in V\}$.

The bdle $c^*\left(\gamma^k(\mathbb{R}^{n+k})\right) \downarrow Gr_n\mathbb{R}^{n+k}$ is denoted by $\gamma_n(\mathbb{R}^{n+k})^{\perp}$.

In order to compute $\pi_* V_n^o(\mathbb{R}^{n+k})$, we needed the LES in htpy associated to a fiber bdle. This sequence is best constructed more generally:

<u>Def'n</u>: (Serre) A <u>fibration</u> is a map $\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ satisfying the <u>homotopy lifting property</u>: given any (solid) diagram

$$\begin{array}{ccc} S^n \times \{0\} & \longrightarrow & E \\ \downarrow & \overset{\tilde{H}}{\nearrow} & \downarrow p \\ S^n \times I & \xrightarrow{H} & B \end{array}$$

there exists a lifting $\tilde{H}$ of the htpy H, making the diagram commute.

<u>Lemma</u> (Serre) Every fiber bdle $\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ is a fibration.

Rmks: These fibrations are often called *Serre* fibrations. Any Serre fibration actually satisfies the htpy lifting property with S^n replaced by *any* CW cplx.

If $\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}$ satisfies the htpy lifting property with S^n replaced by *any* space, then p is called a *Hurewicz Fibration*. Fiber bdles over reasonable base spaces are Hurewicz Fibrations (see [Spanier]).

Fibrations are important because of the following result:

Theorem: Say $\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}$ is a (Serre) fibration, and let $e_0 \in E$ by any basept. Define $p(e_0) =: b_0$, and let $F = p^{-1}(b_0)$ denote the Fiber over b_0. Then there are homomorphisms $\partial : \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ forming a LES

$$\cdots \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \xrightarrow{i_*}$$

(where $i : F \hookrightarrow E$ is the inclusion).

Proof of Lemma: Let $\{U_i\}_{i \in I}$ be an open cover of B over which E is trivial. Given a htpy $H : S^n \times I \to B$ and a lifting $F : S^n \times 0 \to E$ of H_0, choose a triangulation of S^n (via subdivision, say) such that for each simplex

$\sigma \subseteq S^n$, $\exists\, i$ s.t. $H_0(\sigma) \subseteq U_i$. (This is possible b/c the cover $\{H_0^{-1}(U_i)\}_i$ of S^n has positive Lebesgue number, being an open cover of a cpt. set.) We now proceed by induction over the skeleta of S^n. On the zero skeleton, we must extend diagrams

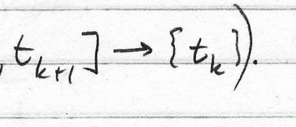
$$\begin{array}{ccc} \{0\} & \xrightarrow{h_0} & E \\ \delta\downarrow & \overset{\tilde{h}}{\nearrow} & \downarrow \\ I & \xrightarrow{h} & B \end{array}$$

Choose $0=t_1<\cdots<t_n=1 \in [0,1]$ s.t. each $[t_i,t_{i+1}]$ lies inside $h^{-1}(U_i)$ for some i, and assume we have defined $\tilde{h}$ on $[0,t_k]$. Then on $[t_k,t_{k+1}]$, we know that

$$\begin{array}{ccc} E|_{h([t_k,t_{k+1}])} & \xleftarrow[\cong]{\varphi_k} & [t_k,t_{k+1}]\times F \\ \downarrow & \swarrow & \\ h([t_k,t_{k+1}]) & & \end{array}$$

, and we set $\tilde{h}(s) = \varphi_k(s, \pi_2\varphi_k^{-1}(h(t_k)))$ (in other words, we extend $\tilde{h}$ via the retraction $[t_k,t_{k+1}]\to\{t_k\}$).

For the inductive step, we assume $\tilde{h}$ is defined on $(k\text{-skeleton of } S^n)\times I$. Given a $(k+1)$-cell $\sigma \subseteq S^n$, we need to extend over $\sigma\times I$. As before we choose $t_i \in I$ s.t. E is trivial over $\sigma\times[t_i,t_{i+1}]$, and we use the retraction $\sigma\times[t_i,t_{i+1}] \longrightarrow \sigma\times\{t_i\} \cup \partial\sigma\times[t_i,t_{i+1}]$, given by "stereographic projection":



. $\square$

Rmk: More generally, this shows that fiber bdles have the HLP for
$$\begin{array}{ccc} X\times 0\cup A\times I & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ X\times I & \longrightarrow & B \end{array}$$
if $A\subseteq X$ is a subcplx $\square$

The LES of a fib'n comes directly out of the htpy lifting property.

<u>Pf of Theorem:</u> We begin by checking exactness of

$\pi_n F \xrightarrow{i_*} \pi_n E \xrightarrow{p_*} \pi_n B$. Since pi is constant,

$p_* i_* = 0$. On the other hand, if $\alpha : S^n \to E$ and

$p_*(\alpha) = 0$, then we have a htpy $S^n \times I \xrightarrow{H} B$ with

$H(* \times I \cup S^n \times \{1\}) = b_0$ ($* \in S^n$ is the basepoint). This

gives a diagram

$$\begin{array}{ccc} S^n \times 0 \cup * \times I & \xrightarrow{\alpha \cup c_{e_0}} & E \\ \downarrow & \overset{\tilde{H}}{\nearrow} & \downarrow p \\ S^n \times I & \xrightarrow{H} & B \end{array}$$

(c_{e_0}: constant map at e_0) so a lift $\tilde{H}$ exists.

Since $p \circ \tilde{H}_1 = H_1 = c_{b_0}$ (the constant map at b_0) we see

that $\tilde{H}_1 : S^n \to E$ actually lands in F, and $[\tilde{H}_1] \in \pi_n F$

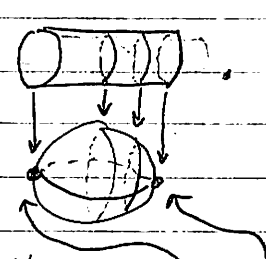
satisfies $i_*[\tilde{H}_1] = [\alpha]$ (b/c $\tilde{H}$ is a htpy from α to $\tilde{H}_1$).

Next we must construct the boundary map $\partial : \pi_n B \to \pi_{n-1} E$.

We begin with two observations regarding spheres. First,

we have a homeomorphism $S^{n-1} \times I / _{S^{n-1} \times 0,\ S^{n-1} \times 1} \cong S^n$,

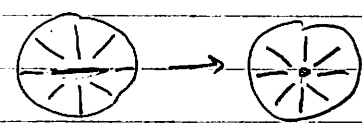
given by vertical projection:



Next,

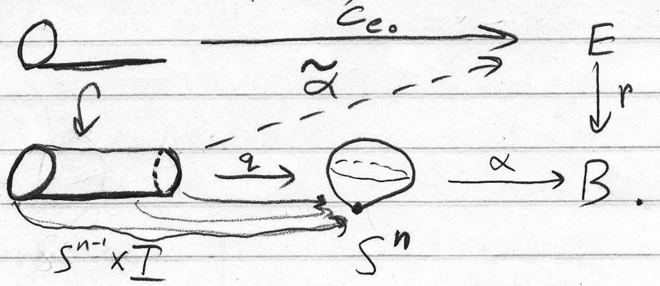
if $I \subset S^n$ denotes an arc between the pts and, then $S^n \cong S^n / I$:

the homeomorphism descends from the map



which collapses an interval in D^n while fixing the bdry.

Now, given $\alpha : S^n \to B$, we consider the diagram



$(\bigstar)$

We know a lift $\tilde{\alpha}$ exists, and $\tilde{\alpha}_1 : S^{n-1} \to E$ actually lands in $F = p^{-1}(b_0)$ b/c $q(S^{n-1} \times \{1\}) = * \in S^n$.

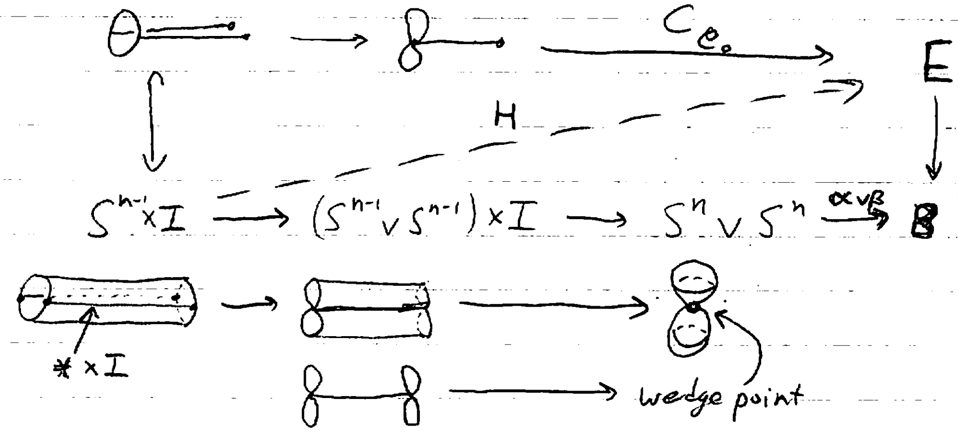
We define $\partial[\alpha] = [\tilde{\alpha}_1]$. We must check that this is a well-defined homomorphism.

<u>Well-defined</u>: Say $H : S^n \times I \to B$ is a htpy from α to β (with $H(* \times I) = b_0$), and say $\tilde{\alpha}$ and $\tilde{\beta}$ are lifts as in diagram $(\bigstar)$. We must show that $\tilde{\alpha} \simeq \tilde{\beta}$. Consider the diagram

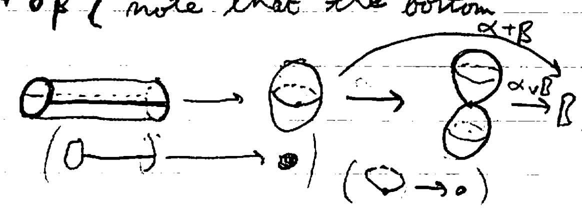
$$\begin{array}{ccc} S^{n-1} \times \{0\} \times I \cup S^{n-1} \times I \times \{0,1\} \cup \{*\} \times I \times I & \xrightarrow{\tilde{\alpha} \cup \tilde{\beta} \cup c_{e_0} \cup c_{e_0}} & E \\ \downarrow & \overset{\tilde{H}}{\nearrow} & \downarrow \\ (S^{n-1} \times I) \times I \xrightarrow{q \times id_I} S^n \times I & \xrightarrow{H} & B. \end{array}$$

The lift $\tilde{H}$ exists, b/c we can apply the HLP to the pair $(S^{n-1} \times \{0\} \times I,\ S^{n-1} \times \{0\} \times \{0,1\} \cup * \times 0 \times I)$, treating the <u>first</u> I coordinate as the htpy coordinate. We now see that $\tilde{H}_1 : S^{n-1} \times \{1\} \times I \to E$ lies in F, and is a htpy from $\tilde{\alpha}_1$ to $\tilde{\beta}_1$ (preserving the basept).

Homomorphism: Given $\alpha, \beta : S^n \to B$, we must find a htpy b/w $\partial\alpha + \partial\beta$ and $\partial(\alpha+\beta)$. Note that since ∂ is well-defined, we can form these elements using any lifts we like. Consider the diagram



The map H factors through $(S^{n-1} \vee S^{n-1}) \times I$, and at time 1 it is both $\partial(\alpha+\beta)$ and $\partial\alpha + \partial\beta$ (note that the bottom map in the diagram factors through



).

Exactness of $\pi_n B \xrightarrow{\partial} \pi_{n-1} F \xrightarrow{i_*} \pi_{n-1} E$ and $\pi_n E \xrightarrow{p_*} \pi_n B \xrightarrow{\partial} \pi_{n-1} F$ are exercises of a similar nature. □

This completes the proof that the Stiefel mflds are universal bdles over the Grassmannians.

We now turn to the construction of important characteristic classes.