The proof of surjectivity of $K(X)\otimes \mathbb{Z}[H]/(H-1)^2 \longrightarrow K(X\times S^2)$ will essentially be complete once we prove:

<u>Prop 2.7</u>: Given a bdle $[E, \overbrace{a(x)z+b(x)}^{\text{linear clutching fcn}}]$, there is a Whitney sum decomposition $E \cong E_+ \oplus E_-$ such that

$$[E, a(x)z+b(x)] \cong [E_+, \mathbb{1}] \oplus [E_-, z] \left(= \left[E_+\oplus E_-, \begin{bmatrix} \mathbb{1} & 0 \\ 0 & z \end{bmatrix}\right]\right).$$

<u>Proof</u>: First we eliminate the coefficient $a(x)$ via a htpy of clutching fcns: the homotopy

$$f_t = (1+tz)\left[a(x)\frac{z+t}{1+tz} + b(x)\right]$$

has $f_0 = a(x)z+b(x)$, and is inv'ble for $t<1$:

$(1+tz)\neq 0$, and $\left|\frac{z+t}{1+tz}\right| = \underbrace{|\bar{z}|}_{1}\left|\frac{z+t}{1+tz}\right| = \left|\frac{\bar{z}(z+t)}{1+tz}\right| = \left|\frac{1+t\bar{z}}{1+tz}\right| = 1.$

So for any $t\in[0,1)$, $[E, a(x)z+b(x)] \cong [E, f_t]$. The coeff. of z in f_t is $a(x)+tb(x)$, which is inv'ble for t close to 1: at $t=1\in S^1$ we know $a(x)+b(x)$ is a clutching fcn, and so $\det(a(x)+b(x))$ is bdd from below over all of X. The same is then true for $t\approx 1$. Now, we claim that

$$[E, a'(x)z+b'(x)] \cong [E, z+(a'(x))^{-1}b(x)]$$

if $a'(x)$ is inv'ble. More generally, if $g(x)$ is inv'ble,

$$[E, f(x,z)] \cong [E, g(x)\circ f(x,z)].$$

This is easy to prove explicitly:

$$E\times D_0 \amalg E\times D_\infty \big/ f \xrightarrow{Id \amalg g\times Id_{D_\infty}} E\times D_0 \amalg E\times D_\infty \big/ gf .$$

This is well-defined b/c if $z\in S^1$,

$$\begin{array}{ccc} (e,z) & \longmapsto & (e,z) \\ \wr\!\wr\!\wr & & \wr\!\wr\!\wr \\ (f_z(e),z) & \longmapsto & (g\circ f_z(e),z) \end{array} .$$

Note: It's very important here that $g(x)$ gives a well-defined isom $E\times D_\infty \xrightarrow{\cong} E\times D_\infty$. If g depended on $z\in D_\infty$ (e.g. $g =$ mult'n by z) then g might not be an isom. at all pts in D_∞ (e.g. at $z=0$).

So we have now reduced to bdles of the form $[E, z+b(x)]$.

Note that $b(x): E\to E$ cannot have eigenvalues in S^1, or else $-z+b(x)$ would become singular for those z.

Now we can decompose any fiber E_x as follows:

$$E_x = (E_x)_+ \oplus (E_x)_-$$

sum of generalized $b(x)$-eigenspaces for eigenvalues λ w/ $|\lambda|>1$

sum of generalized eigenspaces for $|\lambda|<1$

<u>Lemma 2.8:</u> The decompositions $E_x = (E_x)_+ \oplus (E_x)_-$ give rise to a Whitney sum decomposition
$$E = E_+ \oplus E_-$$

Assuming this, we complete the proof of 2.7.

On any fiber E_x, the map $b(x)$ preserves generalized eigenspaces (by def'n) so $b(x)$ sends $(E_x)_+ \to (E_x)_+$
$(E_x)_- \to (E_x)_-$.

Thus $b(x) = b_+(x) \oplus b_-(x)$ and
$$[E, z+b(x)] = \left[E_+ \oplus E_-, \begin{bmatrix} z+b_+(x) & 0 \\ 0 & z+b_-(x) \end{bmatrix}\right]$$
$$= [E_+, z+b_+(x)] \oplus [E_-, z+b_-(x)]$$

But $b_-(x)$ has all eigenvalues inside the unit disk, so $z+t\,b_-(x)$ is inv'ble for any $z \in S^1$, $t \in [0,1]$, and we get a htpy of clutching fcns $z+b_-(x) \simeq z$.

On the other hand, $b_+(x)$ has all eigenvalues <u>outside</u> the unit disk, so $zt + b_+(x)$ is always inv'ble ($t \in [0,1]$).

$$\text{So } [E_+, z+b_+(x)] \cong [E_+, b_+(x)] \underset{\text{multiply by } (b_+(x))^{-1}}{\cong} [E_+, \mathbb{1}]$$
and $[E_-, z+b_-(x)] \cong [E_-, z]$.

Thus we've shown $[E, a(x)z + b(x)] \cong [E_+, \mathbb{1}] \oplus [E_-, z]$ as desired. $\square$